Large deviations and support results for nonlinear Schrödinger equations with additive noise and applications

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LARGE DEVIATIONS AND SUPPORT RESULTS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH ADDITIVE NOISE AND APPLICATIONS

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Abstract. Sample path large deviations for the laws of the solutions of stochastic nonlinear Schrödinger equations when the noise converges to zero are presented. The noise is a complex additive gaussian noise. It is white in time and colored space wise. The solutions may be global or blow-up in finite time, the two cases are distinguished. The results are stated in trajectory spaces endowed with projective limit topologies. In this setting, the support of the law of the solution is also characterized. As a consequence, results on the law of the blow-up time and asymptotics when the noise converges to zero are obtained. An application to the transmission of solitary waves in fiber optics is also given.

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1. Introduction

In the present article, the stochastic nonlinear Schrödinger (NLS) equation with a power law nonlinearity and an additive noise is studied. The deterministic equation occurs as a basic model in many areas of physics: hydrodynamics, plasma physics, nonlinear optics, molecular biology. It describes the propagation of waves in media with both nonlinear and dispersive responses. It is an idealized model and does not take into account many aspects such as inhomogeneities, high order terms, thermal fluctuations, external forces which may be modeled as a random excitation (see [3, 4, 16, 18]). Propagation in random media may also be considered. The resulting re-scaled equation is a random perturbation of the dynamical system of the following form:

$$i\frac{\partial}{\partial t}\psi - (\Delta\psi + \lambda|\psi|^{2\sigma}\psi) = \xi, \quad x \in \mathbb{R}^d, \quad t \geq 0, \quad \lambda = \pm 1, \tag{1.1}$$

where ξ is a complex valued space-time white noise with correlation function, following the denomination used in [16],

$$\mathbb{E}\left[\xi(t_1, x_1)\bar{\xi}(t_2, x_2)\right] = D\delta_{t_1 - t_2} \otimes \delta_{x_1 - x_2}$$

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D is the noise intensity and δ denotes the Dirac mass. When $\lambda = 1$ the nonlinearity is called focusing, otherwise it is defocusing.

With the notations of section 2, the unbounded operator $-i\Delta$ on $\mathrm{L}^2(\mathbb{R}^d)$ with domain $\mathrm{H}^2(\mathbb{R}^d)$ is skew-adjoint. Stone's theorem gives thus that it generates a unitary group $(S(t) = e^{-it\Delta})_{t\in\mathbb{R}}$. The Fourier transform gives that this group is also unitary on every Sobolev space based on $\mathrm{L}^2(\mathbb{R}^d)$. Consequently, there is no smoothing effect in the Sobolev spaces.

We are thus unable to treat the space-time white noise and will consider a complex valued centered gaussian noise, white in time and colored space wise.

In the present article, the formalism of stochastic evolution equations in Banach spaces as presented in [8] is adopted. This point of view is preferred to the field and martingale measure stochastic integral approach, see [20], in order to use a particular property of the group, namely hyper-contractivity. The Strichartz estimates, presented in the next section, show that some integrability property is gained through time integration and "convolution" with the group. In this setting, the gaussian noise is defined as the time derivative in the sense of distributions of a Q-Wiener process $(W(t))_{t\in[0,+\infty)}$ on $\mathrm{H}^1(\mathbb{R}^d)$. Here Q is the covariance operator of the law of the $\mathrm{H}^1(\mathbb{R}^d)-$random variable $W(1)$, which is a centered gaussian measure. With the Itô notations, the stochastic evolution equation is written

$$i\mathrm{d}u - (\Delta u + \lambda |u|^{2\sigma} u)\mathrm{d}t = \mathrm{d}W. \tag{1.2}$$

The initial datum u_0 is a function of $\mathrm{H}^1(\mathbb{R}^d)$. We will consider solutions of NLS that are weak solutions in the sense used in the analysis of partial differential equations or equivalently mild solutions which satisfy

$$u(t) = S(t)u_0 - i\lambda \int_0^t S(t-s)(|u(s)|^{2\sigma}u(s))ds - i\int_0^t S(t-s)dW(s). \tag{1.3}$$

The well posedness of the Cauchy problem associated to (1.1) in the deterministic case depends on the size of σ. If $\sigma < \frac{2}{d}$, the nonlinearity is subcritical and the Cauchy problem is globally well posed in $\mathrm{L}^2(\mathbb{R}^d)$ or $\mathrm{H}^1(\mathbb{R}^d)$. If $\sigma = \frac{2}{d}$, critical nonlinearity, or $\frac{2}{d} < \sigma < \frac{2}{d-2}$ when $d \geq 3$ or simply $\sigma > \frac{2}{d}$ otherwise, supercritical nonlinearity, the Cauchy problem is locally well posed in $\mathrm{H}^1(\mathbb{R}^d)$, see [17]. In this latter case, if the nonlinearity is defocusing, the solution is global. In the focusing case, when the nonlinearity is critical or supercritical, some initial data yield global solutions while it is known that other initial data yield solutions which blow up in finite time, see [7, 19].

In [9], the $\mathrm{H}^1(\mathbb{R}^d)$ results have been generalized to the stochastic case and existence and uniqueness results are obtained for the stochastic equation under the same conditions on σ. Continuity with respect to the initial data and the perturbation is proved. It is shown that the proof of global existence for a defocusing nonlinearity or for a focusing nonlinearity with a subcritical exponent, could be adapted in the stochastic case even if the momentum

$$M(u) = \|u\|_{\mathrm{L}^2(\mathbb{R}^d)}$$

and hamiltonian

$$H(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx - \frac{\lambda}{2\sigma+2}\int_{\mathbb{R}^d} |u|^{2\sigma+2} dx$$

are no longer conserved. For a focusing nonlinearity and critical or supercritical exponents, the solution may blow-up in finite time. The blow-up time is denoted by $\tau(\omega)$. It satisfies either $\lim_{t\to\tau(\omega)} \|u(t)\|_{\mathrm{H}^1(\mathbb{R}^d)} = +\infty$ or $\tau(\omega) = +\infty$, even if the solution is obtained by a fixed point argument in a ball of a space of more regular functions than $\mathrm{C}([0,T];\mathrm{H}^1(\mathbb{R}^d))$.

In this article, we are interested in the law of the paths of the random solution. When the noise converges to zero, continuity with respect to the perturbation gives that the law converges to the Dirac mass on the deterministic solution. In the following, a large deviation result is shown. It gives the rate of convergence to zero, on an exponential scale, of the probability that paths are in sets that do not contain the deterministic solution. A general result is stated for the case where blow-up in finite time is possible and a second one for the particular case where the solutions are global. Also, the stronger is the topology, the sharper are the estimates. We will therefore take advantage of the variety of spaces that can be considered for the fixed point argument, due to the integrability property, and present the large deviation principles in trajectory spaces endowed with projective limit topologies. A characterization of the support of the law of the solution in these trajectory spaces is proved. The two results can be transferred to weaker topologies or more generally by any continuous mapping. The first application is a proof that, for certain noises, with positive probability some solutions blow up after any time T. Some estimates on the law of the blow-up time when the noise converges to zero are also obtained. This study is yet another contribution to the study of the influence of a noise on the blow-up of the solutions of the focusing supercritical NLS, see in the case of an additive noise [10, 11]. A second application is given. It consists in obtaining similar results as in [16] with an approach based on large deviations. The aim is to compute estimates of error probability in signal transmission in optical fibers when the medium is random and nonlinear, for small noises.

Section 2 is devoted to notations and properties of the group, of the noise and of the stochastic convolution. An extension of the result of continuity with respect to the stochastic convolution presented in [9] is also given. In section 3, the large deviation principles (LDP) is presented. Section 4 is devoted to the support result and the two last sections to the applications.

2. NOTATIONS AND PRELIMINARY RESULTS

Throughout the paper the following notations will be used.

For $p \in \mathbb{N}^*$, $\mathrm{L}^p(\mathbb{R}^d)$ is the classical Lebesgue space of complex valued functions and $\mathrm{W}^{1,p}(\mathbb{R}^d)$ is the associated Sobolev space of $\mathrm{L}^p(\mathbb{R}^d)$ functions with first order derivatives, in the sense of distributions, in $\mathrm{L}^p(\mathbb{R}^d)$. When $p = 2$, $\mathrm{H}^s(\mathbb{R}^d)$ denotes the fractional Sobolev space of tempered distributions $v \in \mathcal{S}'(\mathbb{R}^d)$ such that the Fourier transform $\hat{v}$ satisfies $(1 + |\xi|^2)^{s/2}\hat{v} \in \mathrm{L}^2(\mathbb{R}^d)$. The space $\mathrm{L}^2(\mathbb{R}^d)$ is endowed with the inner product defined by $(u,v)_{\mathrm{L}^2(\mathbb{R}^d)} = \Re \int_{\mathbb{R}^d} u(x)\overline{v}(x)dx$. Also, when it is clear that μ is a Borel measure on a specified Banach space, we simply write $\mathrm{L}^2(\mu)$ and do not specify the Banach space and Borel $\sigma-$field.

If I is an interval of $\mathbb{R}$, $(E, \|\cdot\|_E)$ a Banach space and r belongs to $[1,+\infty]$, then $\mathrm{L}^r(I;E)$ is the space of strongly Lebesgue measurable functions f from I into E such that $t \to \|f(t)\|_E$ is in $\mathrm{L}^r(I)$. Let $\mathrm{L}^r_{loc}(0,+\infty;E)$ be the respective spaces of locally integrable functions on $(0,+\infty)$. They are endowed with topologies of Fréchet space. The spaces $\mathrm{L}^r(\Omega;E)$ are defined similarly.

We recall that a pair (r,p) of positive numbers is called an admissible pair if p satisfies $2\leq p<\frac{2d}{d-2}$ when $d>2$ ($2\leq p<+\infty$ when $d=2$ and $2\leq p\leq+\infty$ when $d=1$) and r is such that $\frac{2}{r}=d\left(\frac{1}{2}-\frac{1}{p}\right)$. For example $(+\infty,2)$ is an admissible pair.

When E is a Banach space, we will denote by E^* its topological dual space. For $x^*\in E^*$ and $x\in E$, the duality will be denoted $<x^*,x>_{E^*,E}$.

We recall that Φ is a Hilbert Schmidt operator from a Hilbert space H into a Hilbert space $\tilde{H}$ if it is a linear continuous operator such that, given a complete orthonormal system $(e_j^H)_{j\in\mathbb{N}}$ of H, $\sum_{j\in\mathbb{N}}\|\Phi e_j^H\|_{\tilde{H}}^2<+\infty$. We will denote by $\mathcal{L}_2(H,\tilde{H})$ the space of Hilbert Schmidt operators from H into $\tilde{H}$ endowed with the norm

$$\|\Phi\|_{\mathcal{L}_2(H,\tilde{H})}=\mathrm{tr}\left(\Phi\Phi^*\right)=\sum_{j\in\mathbb{N}}\|\Phi e_j^H\|_{\tilde{H}}^2,$$

where Φ^* denotes the adjoint of Φ and tr the trace. We denote by $\mathcal{L}_2^{s,r}$ the corresponding space for $H=\mathrm{H}^s(\mathbb{R}^d)$ and $\tilde{H}=\mathrm{H}^r(\mathbb{R}^d)$. In the introduction Φ has been taken in $\mathcal{L}_2^{0,1}$.

When A and B are two Banach spaces, $A\cap B$, where the norm of an element is defined as the maximum of the norm in A and in B, is a Banach space. The following Banach spaces defined for the admissible pair $(r(p),p)$ and positive T by

$$X^{(T,p)}=\mathrm{C}\left([0,T];\mathrm{H}^1(\mathbb{R}^d)\right)\cap\mathrm{L}^{r(p)}\left(0,T;\mathrm{W}^{1,p}(\mathbb{R}^d)\right)$$

will be of particular interest.

The two following Hilbert spaces of spatially localized functions are also introduced,

$$\Sigma=\{f\in\mathrm{H}^1(\mathbb{R}^d):x\mapsto|x|f(x)\in\mathrm{L}^2(\mathbb{R}^d)\}$$

endowed with the norm $\|\cdot\|_\Sigma$ defined by

$$\|f\|_\Sigma^2=\|f\|_{\mathrm{H}^1(\mathbb{R}^d)}^2+\|x\mapsto|x|f(x)\|_{\mathrm{L}^2(\mathbb{R}^d)}^2,$$

and $\Sigma^{\frac{1}{2}}$ where $|x|$ is replaced by $\sqrt{|x|}$. The variance is defined as the quantity

$$V(f)=\int_{\mathbb{R}^d}|x|^2|f(x)|^2dx,\ f\in\Sigma.$$

Also we denote by $eval_x(f)$ the evaluation of a function f at the point x where f is a function taking value in any topological space.

The probability space will be denoted by $(\Omega,\mathcal{F},\mathbb{P})$. Also, $x\wedge y$ stands for the minimum of the two real numbers x and y and $x\vee y$ for the maximum. We recall that a rate function I is a lower semicontinuous function and that a good rate function I is a rate function such that for every $c>0$, $\{x:I(x)\leq c\}$ is a compact set. Finally, we will denote by *supp* μ the support of a probability measure μ on a topological vector space. It is the complementary of the largest open set of null measure.

2.1. **Properties of the group.** When the group acts on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, the Fourier transform gives the following analytic expression

$$\forall u_0 \in \mathcal{S}(\mathbb{R}^d),\ \forall t \neq 0,\ S(t)u_0 = \frac{1}{(4i\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\frac{|x-y|^2}{4t}} u_0(y)dy.$$

The Fourier transform also gives that the adjoint of $S(t)$ in $\mathrm{L}^2(\mathbb{R}^d)$ and in every Sobolev space on $\mathrm{L}^2(\mathbb{R}^d)$ is $S(-t)$, the same bounded operator with time reversal.

The Strichartz estimates, see [17], are the following

i/ $\forall\ u_0 \in \mathrm{L}^2(\mathbb{R}^d),\ \forall\ (r,p)$ admissible pair,

$$t \mapsto S(t)u_0 \in \mathrm{C}(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^d)) \cap \mathrm{L}^r(\mathbb{R}; \mathrm{L}^p(\mathbb{R}^d)),$$

and there exists a positive constant c such that,

$$\|S(\cdot)u_0\|_{\mathrm{L}^r(\mathbb{R};\mathrm{L}^p(\mathbb{R}^d))} \leq c\|u_0\|_{\mathrm{L}^2(\mathbb{R}^d)}.$$

ii/ For $T > 0$, for all $(r(p), p)$ and $(r(q), q)$ two admissible pairs, if s and ρ are the conjugate exponents of $r(q)$ and q, i.e. $\frac{1}{s} + \frac{1}{r(q)} = 1$ and $\frac{1}{q} + \frac{1}{\rho} = 1$,

$$\forall f \in \mathrm{L}^s(0,T; \mathrm{L}^\rho(\mathbb{R}^d)),\ \Lambda f \in\ \mathrm{C}([0,T]; \mathrm{L}^2(\mathbb{R}^d)) \cap \mathrm{L}^{r(p)}(0,T; \mathrm{L}^p(\mathbb{R}^d))$$

where Λ is defined by $\Lambda f = \int_0^{\cdot} S(\cdot - s)f(s)ds$. Moreover, Λ is a continuous linear operator from $\mathrm{L}^s(0,T; \mathrm{L}^\rho(\mathbb{R}^d))$ into $\mathrm{L}^{r(p)}(0,T; \mathrm{L}^p(\mathbb{R}^d))$ with a norm that does not depend on T.

Remark 2.1. The first estimate gives the integrability property of the group, the second gives the integrability of the convolution that allows to treat the nonlinearity.

2.2. **Topology and trajectory spaces.** Let us introduce a topological space that allows to treat the subcritical case or the defocusing case. When $d > 2$, we set

$$\mathcal{X}_\infty = \bigcap_{T \in \mathbb{R}_+^*,\ 2 \leq p < \frac{2d}{d-2}} X^{(T,p)},$$

it is endowed with the projective limit topology, see [5] and [14]. When $d = 2$ and $d = 1$ we write $p \in [2, +\infty)$.

The set of indices $(J, \prec)$, where $(T,p) \prec (S,q)$ if $T \leq S$ and $p \leq q$, is a partially ordered right-filtering set.

If $(T,p) \prec (S,q)$ and $u \in X^{(S,q)}$, Hölder's inequality gives that for α such that $\frac{1}{p} = \frac{\alpha}{q} + \frac{1-\alpha}{2}$,

$$\exists\ c(p,q) > 0 :\ \|u(t)\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq c(p,q)\|u(t)\|_{\mathrm{L}^2(\mathbb{R}^d)}^{1-\alpha}\|u(t)\|_{\mathrm{L}^q(\mathbb{R}^d)}^{\alpha}.$$

Consequently,

$$\|u(t)\|_{\mathrm{W}^{1,p}(\mathbb{R}^d)} \leq (d+1)c(p,q)\|u(t)\|_{\mathrm{H}^1(\mathbb{R}^d)}^{1-\alpha}\|u(t)\|_{\mathrm{W}^{1,q}(\mathbb{R}^d)}^{\alpha}.$$

By time integration, along with Hölder's inequality and the fact that $\alpha r(q) = r(p)$, u is a function of $X^{(T,p)}$ and

$$\|u\|_{X^{(T,p)}} \leq (d+1)c(p,q)\|u\|_{X^{(S,q)}}. \tag{2.1}$$

If we denote by $p_{(S,q)}^{(T,p)}$ the dense and continuous embeddings from $X^{(S,q)}$ into $X^{(T,p)}$, they satisfy the consistency conditions

$$\forall\ (T,p) \prec (S,q) \prec (R,r),\ p_{(R,r)}^{(T,p)} = p_{(R,r)}^{(S,q)} \circ p_{(S,q)}^{(T,p)}.$$

Consequently, the projective limit topology is well defined by the following neighborhood basis, given for φ_1 in $\mathcal{X}_\infty$ by

$$U(\varphi_1;(T,p);\epsilon) = \left\{ \varphi \in \bigcap_{(T',p')\in J} X^{(T',p')} : \|\varphi - \varphi_1\|_{X^{(T,p)}} < \epsilon \right\}.$$

It is the weakest topology on the intersection such that for every $(T,p) \in J$, the injection $p_{(T,p)} : \mathcal{X}_\infty \to X^{(T,p)}$ is continuous. It is a standard fact, see [5], that $\mathcal{X}_\infty$ is a Hausdorff topological space.
Following from (2.1), a countable neighborhood basis of φ_1 is given by

$$\left(U\left(\varphi_1; (n, p(l)); \frac{1}{k} \right) \right)_{(n,k,l)\in(N^*)^3},$$

where $p(l) = 2 + \frac{4}{d-2} - \frac{1}{l}$ and $l > \frac{d-2}{4}$ if $d > 2$. If $d = 2$ and $d = 1$, we take $p(l) = l$. It is convenient, for measurability issues, to introduce the countable metric version

$$\mathcal{D} = \bigcap_{n\in\mathbb{N}^*:\ n>\frac{d-2}{4}} X^{(n,p(n))};\ \forall (x,y) \in \mathcal{D}^2,\ d(x,y) = \sum_{n>\frac{d-2}{4}} \frac{1}{2^n} \left(\|x-y\|_{X^{(n,p(n))}} \vee 1 \right).$$

It is classical that $\mathcal{D}$ is a complete separable metric space, i.e. a Polish space. The previous discussion gives that $\mathcal{D}$ is homeomorphic to $\mathcal{X}_\infty$. Remark that it can be checked that $\mathcal{D}$ is a locally convex Fréchet space.

The following spaces are introduced for the case where blow-up may occur. Adding a point Δ to the space $\mathrm{H}^1(\mathbb{R}^d)$ and adapting slightly the proof of Alexandroff's compactification, it can be seen that the open sets of $\mathrm{H}^1(\mathbb{R}^d)$ and the complementary in $\mathrm{H}^1(\mathbb{R}^d) \cup \{\Delta\}$ of the closed bounded sets of $\mathrm{H}^1(\mathbb{R}^d)$ define the open sets of a topology on $\mathrm{H}^1(\mathbb{R}^d) \cup \{\Delta\}$. This topology induces on $\mathrm{H}^1(\mathbb{R}^d)$ the topology of $\mathrm{H}^1(\mathbb{R}^d)$. Also, with such a topology $\mathrm{H}^1(\mathbb{R}^d) \cup \{\Delta\}$ is a Hausdorff topological space. Remark that in [1], where diffusions are studied, the compactification of $\mathbb{R}^d$ is considered. Nonetheless, compactness is not an important feature and the above construction is enough for the following.
The space $\mathrm{C}([0,+\infty); \mathrm{H}^1(\mathbb{R}^d) \cup \{\Delta\})$ is the space of continuous functions with value in $\mathrm{H}^1(\mathbb{R}^d) \cup \{\Delta\}$. Also, if f belongs to $\mathrm{C}([0,+\infty); \mathrm{H}^1(\mathbb{R}^d) \cup \{\Delta\})$ we denote the blow-up time by

$$\mathcal{T}(f) = \inf\{t \in [0,+\infty) : f(t) = \Delta\}.$$

As in [1], a space of exploding paths, where Δ acts as a cemetery, is introduced. We set

$$\mathcal{E}(\mathrm{H}^1(\mathbb{R}^d)) = \left\{ f \in \mathrm{C}([0,+\infty); \mathrm{H}^1(\mathbb{R}^d) \cup \{\Delta\}) : f(t_0) = \Delta \Rightarrow \forall t \geq t_0,\ f(t) = \Delta \right\}.$$

It is endowed with the topology defined by the following neighborhood basis given for φ_1 in $\mathcal{E}(\mathrm{H}^1(\mathbb{R}^d))$ by

$$V_{T,\ \epsilon}(\varphi_1) = \left\{ \varphi \in \mathcal{E}(\mathrm{H}^1(\mathbb{R}^d)) : \mathcal{T}(\varphi) \geq \mathcal{T}(\varphi_1),\ \|\varphi_1 - \varphi\|_{\mathrm{L}^\infty([0,T];\mathrm{H}^1(\mathbb{R}^d))} \leq \epsilon \right\},$$

where $T < \mathcal{T}(\varphi_1)$ and $\epsilon > 0$.
As a consequence of the topology of $\mathcal{E}(\mathrm{H}^1(\mathbb{R}^d))$, the function $\mathcal{T}$ from $\mathcal{E}(\mathrm{H}^1(\mathbb{R}^d))$

into $[0,+\infty]$ is sequentially lower semicontinuous, this is to say that if a sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges to f then $\underline{\lim}_{n\to+\infty}\mathcal{T}(f_n)\geq\mathcal{T}(f)$. Following from (2.1), the topology of $\mathcal{E}(\mathrm{H}^1(\mathbb{R}^d))$ is also defined by the countable neighborhood basis given for $\varphi_1\in\mathcal{E}(\mathrm{H}^1(\mathbb{R}^d))$ by $\left(V_{\mathcal{T}(\varphi_1)-\frac{1}{n},\frac{1}{k}}(\varphi_1)\right)_{(n,k)\in(\mathbb{N}^*)^2}$. Therefore $\mathcal{T}$ is a lower semicontinuous mapping.
Remark that, as topological spaces, the two following spaces satisfy the identity

$$\left\{f\in\mathcal{E}(\mathrm{H}^1(\mathbb{R}^d)):\mathcal{T}(f)=+\infty\right\}=\mathrm{C}([0,+\infty);\mathrm{H}^1(\mathbb{R}^d)).$$

Finally, the analogous of the intersection in the subcritical case endowed with projective limit topology $\mathcal{E}_\infty$ is defined, when $d>2$, by

$$\left\{f\in\mathcal{E}(\mathrm{H}^1(\mathbb{R}^d)):\forall\, p\in\left[2,\frac{2d}{d-2}\right),\ \forall\, T\in[0,\mathcal{T}(f)),\ f\in\mathrm{L}^{r(p)}\left(0,T;\mathrm{W}^{1,p}(\mathbb{R}^d)\right)\right\}.$$

When $d=2$ and $d=1$ we write $p\in[2,+\infty)$. It is endowed with the topology defined for φ_1 in $\mathcal{E}_\infty$ by the following neighborhood basis

$$W_{T,p,\ \epsilon}(\varphi_1)=\left\{\varphi\in\mathcal{E}_\infty:\mathcal{T}(\varphi)\geq\mathcal{T}(\varphi_1),\ \|\varphi_1-\varphi\|_{X^{(T,p)}}\leq\epsilon\right\}.$$

where $T<\mathcal{T}(\varphi_1)$, p is as above and $\epsilon>0$. From the same arguments as for the space $\mathcal{X}_\infty$, $\mathcal{E}_\infty$ is a Hausdorff topological space. Also, as previously, (2.1) gives that the topology can be defined for φ_1 in $\mathcal{E}_\infty$ by the countable neighborhood basis $\left(W_{\mathcal{T}(\varphi_1)-\frac{1}{n},p(n),\frac{1}{k}}(\varphi_1)\right)_{(n,k)\in(\mathbb{N}^*)^2:\ n>\frac{d-2}{4}}$.

If we denote again by $\mathcal{T}:\mathcal{E}_\infty\to[0,+\infty]$ the blow-up time, since $\mathcal{E}_\infty$ is continuously embedded into $\mathcal{E}(\mathrm{H}^1(\mathbb{R}^d))$, $\mathcal{T}$ is lower semicontinuous. Thus, since the sets $\{[0,t],\ t\in[0,+\infty]\}$ is a $\pi-$system that generates the Borel $\sigma-$algebra of $[0,+\infty]$, $\mathcal{T}$ is measurable. Remark also that, as topological spaces, the following spaces are identical

$$\{f\in\mathcal{E}_\infty:\mathcal{T}(f)=+\infty\}=\mathcal{X}_\infty.$$

2.3. **Statistical properties of the noise.** The Q-Wiener process W is such that its trajectories are in $\mathrm{C}([0,+\infty);\mathrm{H}^1(\mathbb{R}^d))$. We assume that $Q=\Phi\Phi^*$ where Φ is a Hilbert Schmidt operator from $\mathrm{L}^2(\mathbb{R}^d)$ into $\mathrm{H}^1(\mathbb{R}^d)$. The Wiener process can therefore be written as $W=\Phi W_c$ where W_c is a cylindrical Wiener process. Recall that for any orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of $\mathrm{L}^2(\mathbb{R}^d)$, there exists a sequence of real independent Brownian motions $(\beta_j)_{j\in\mathbb{N}}$ such that $W_c=\sum_{j\in\mathbb{N}}\beta_je_j$.

The sum $W_c=\sum_{j\in\mathbb{N}}\beta_je_j$ is well defined in every Hilbert space H such that $\mathrm{L}^2(\mathbb{R}^d)$ is embedded into H with a Hilbert Schmidt embedding. The denomination cylindrical is justified by the fact that the decomposition of $W_c(1)$ on cylinder sets $(e_1,...,e_N)$ are the finite dimensional centered gaussian variables $(\beta_1(1),...,\beta_N(1))$ with a covariance equal to the identity. The natural extension of the corresponding sequence of centered gaussian measures in finite dimensions, with a covariance equal to identity, is a gaussian cylindrical measure. We denote it by ν. The law of $W(1)$ is then the $\sigma-$additive direct image measure $\mu=\Phi_*\nu$.

The Karhunen-Loeve decomposition of the Brownian motions on $\mathrm{L}^2(0,1)$ is the decomposition on the eigenvectors $\left(f_i(t)=\sqrt{2}\sin\left(\left(i+\frac{1}{2}\right)\pi t\right)\right)_{i\in\mathbb{N}}$ of the injective correlation operator $\varphi\mapsto\int_0^1(s\wedge\cdot)\varphi(s)ds$ which form a complete orthonormal system. The coefficients are then independent real-valued centered normal random variables where the variances are equal to the eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$. In addition, $\left(g_i = \sqrt{\lambda_i}\frac{\partial}{\partial t} f_i\right)_{i\in\mathbb{N}}$ also forms a complete orthonormal system of $\mathrm{L}^2(0,1)$. Thus, formally, the coefficients of the series expansion of the derivative of W_c on the tensor product of the complete orthonormal systems are a sequence of independent real-valued standard normal random variables. It is thus a gaussian white noise.

The correlation operator $\tilde{Q}$ of our space-colored noise, for $0 \le t \le t+s \le 1$, $(x,z) \in (\mathbb{R}^d)^2$, $(h,k) \in \mathbb{C}^2$ and $(z,t)_{\mathbb{C}} = \Re(z\bar{t})$, is formally given by

$$\mathbb{E}\left[\left(\frac{\partial W}{\partial t}(t+s,x+z),h\right)_{\mathbb{C}}\left(\frac{\partial W}{\partial t}(t,x),k\right)_{\mathbb{C}}\right] = \left(\tilde{Q}k,h\right)_{\mathbb{C}},$$

where

$$\tilde{Q}k = \sum_{(i,j)\in\mathbb{N}^2} g_i(t+s)g_i(t)\Phi e_j(x+z)\left(\Phi e_j(x),k\right)_{\mathbb{C}}.$$

Testing the distribution on the product of smooth complex-valued functions with compact support on $\mathbb{R}^+$ and $\mathbb{R}^d$, respectively ψ and φ, we get

$$\int_0^1\int_{\mathbb{R}^d}\tilde{Q}k\psi(s)\varphi(z)dsdz = \psi(0)\overline{\Phi\Phi^*\overline{\varphi}(\cdot - x)}(x)k.$$

The correlation operator could then be identified with the multiplication by the distribution $\delta_0 \otimes \overline{eval_x\Phi\Phi^*\overline{\tau_{-x}}}$. Remark that if Φ were the identity map, we would obtain the multiplication by the Dirac mass $\delta_{(0,0)}$.

The operator Φ belongs to $\mathcal{L}_2^{0,1}$ and thus to $\mathcal{L}_2^{0,0}$ and is defined through the kernel $\mathcal{K}(x,y) = \frac{1}{2}\sum_{j\in\mathbb{N}}\Phi\overline{e_j}(x)e_j(y)$ of $\mathrm{L}^2(\mathbb{R}^d\times\mathbb{R}^d)$. This means that for any square integrable function u, $\Phi u(x) = \int_{\mathbb{R}^d}\mathcal{K}(x,y)u(y)dy$. Since Φ is in $\mathcal{L}_2^{0,1}$, the kernel satisfies

$$\|\Phi\|_{\mathcal{L}_2^{0,1}} = \|\mathcal{K}\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathbb{R}^d)} + \sum_{n=1}^d\sum_{j\in\mathbb{N}}\left\|\int_{\mathbb{R}^d}\frac{\partial}{\partial x_n}\mathcal{K}(\cdot,y)e_j(y)dy\right\|^2_{\mathrm{L}^2(\mathbb{R}^d)}.$$

Remark that since it is impossible that $\mathcal{K}(x,y) = \mathcal{K}(x-y)$, the noise is not homogeneous. The distribution $\overline{eval_x\Phi\Phi^*\overline{\tau_{-x}}}$ is then the $\mathrm{L}^2(\mathbb{R}^d)$ function

$$\frac{1}{2}\sum_{j\in\mathbb{N}}\Phi e_j(x+z)\overline{\Phi e_j(x)} \text{ or } \int_{\mathbb{R}^d}\mathcal{K}(x+z,u)\overline{\mathcal{K}}(x,u)du.$$

Some authors use the terminology correlation function as in our introduction. In this setting we obtain

$$\mathbb{E}\left[\frac{\partial}{\partial t}W(t+s,x+z)\overline{\frac{\partial}{\partial t}W(t,x)}\right] = 2\tilde{Q}$$

and

$$\mathbb{E}\left[\frac{\partial}{\partial t}W_c(t_1,x_1)\overline{\frac{\partial}{\partial t}W_c(t_2,x_2)}\right] = 2\delta_{t_1-t_2}\otimes\delta_{x_1-x_2}$$

in the case of white noise.

Finally, recall the more standard result that the correlation operator of the Q-Wiener process is $(t\wedge s)\Phi\Phi^*$.

In the following we assume that the probability space is endowed with the filtration $\mathcal{F}_t = \mathcal{N}\cup\sigma\{W_s, 0\le s\le t\}$ where $\mathcal{N}$ denotes the $\mathbb{P}-$null sets.

2.4. **The random perturbation.** Considering the fixed point problem, the stochastic convolution $Z(t) = \int_0^t S(t-s)dW(s)$ is a stochastic perturbation term. Let us define the operator $\mathcal{L}$ on $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))$ by

$$\mathcal{L}h(t) = \int_0^t I \circ S(t-s)\Phi h(s)ds, \ \ h \in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d)),$$

where I is the injection of $\mathrm{H}^1(\mathbb{R}^d)$ into $\mathrm{L}^2(\mathbb{R}^d)$.

Proposition 2.2. *The stochastic convolution defines a measurable mapping from $(\Omega,\mathcal{F})$ into $\left(\mathcal{X}_\infty, \mathcal{B}^X\right)$, where $\mathcal{B}^X$ stands for the Borel $\sigma-$field. Its law is denoted by μ^Z.*
The direct images $\mu^{Z;(T,p)} = p_{(T,p)*}\mu^Z$ on the real Banach spaces $X^{(T,p)}$ are centered gaussian measures of reproducing kernel Hilbert space (RKHS) $H_{\mu^{Z;(T,p)}} = \mathrm{im}\mathcal{L}$ with the norm of the image structure.

Proof. Setting $F(t) = \int_0^t S(-u)dW(u)$, for $t \in \mathbb{R}^+$, $Z(t) = S(t)F(t)$ follows. Indeed, if $(f_j)_{j\in\mathbb{N}}$ is a complete orthonormal system of $\mathrm{H}^1(\mathbb{R}^d)$, a straightforward calculation gives that $(Z(t), f_j)_{\mathrm{H}^1(\mathbb{R}^d)} = (S(t)F(t), f_j)_{\mathrm{H}^1(\mathbb{R}^d)}$ for every j in $\mathbb{N}$. The continuity of the paths follows from the construction of the stochastic integral of measurable and adapted operator integrands satisfying $\mathbb{E}\left[\int_0^t \|S(-u)\Phi\|^2_{\mathcal{L}_2^{0,0}}\right] < +\infty$ with respect to the Wiener process and from the strong continuity of the group. Consequently, for every positive T, the paths are in $\mathrm{C}([0,T];\mathrm{H}^1(\mathbb{R}^d))$.

Step 1: The mapping Z is measurable from $(\Omega,\mathcal{F})$ into $\left(X^{(T,p)}, \mathcal{B}^{(T,p)}\right)$, where $\mathcal{B}^{(T,p)}$ denotes the associated Borel $\sigma-$field.
Since $X^{(T,p)}$ is a Polish space, every open set is a countable union of open balls and consequently $\mathcal{B}^{(T,p)}$ is generated by open balls.
Remark that the event $\{\omega \in \Omega : \|Z(\omega) - x\|_{X^{(T,p)}} \le r\}$ is the intersection of

$$\bigcap_{s\in\mathbb{Q}\cap[0,T]} \left\{\omega \in \Omega : \|Z(s)(\omega) - x\|_{\mathrm{H}^1(\mathbb{R}^d)} \le r\right\}$$

and of

$$\left\{\omega \in \Omega : \|Z(\omega) - x\|_{\mathrm{L}^{r(p)}(0,T;\mathrm{W}^{1,p}(\mathbb{R}^d))} \le r\right\}.$$

Also, remark that, since $(Z(t))_{t\in\mathbb{R}^+}$ is a collection of $\mathrm{H}^1(\mathbb{R}^d)$ random variables, the first part is a countable intersection of elements of $\mathcal{F}$. Consequently, it suffices to show that : $\omega \mapsto (t \mapsto Z(t))$ defines a $\mathrm{L}^{r(p)}(0,T;\mathrm{W}^{1,p}(\mathbb{R}^d))$ random variable.
Consider $(\Phi_n)_{n\in\mathbb{N}}$ a sequence of operators of $\mathcal{L}_2^{0,2}$ converging to Φ for the topology of $\mathcal{L}_2^{0,1}$ and Z_n the associated stochastic convolutions. The Sobolev injections along with Hölder's inequality give that when $d > 2$ and $2 \le p \le \frac{2d}{d-2}$, $\mathrm{H}^1(\mathbb{R}^d)$ is continuously embedded in $\mathrm{L}^p(\mathbb{R}^d)$. It also gives that, when $d = 2$, $\mathrm{H}^1(\mathbb{R}^d)$ is continuously embedded in every $\mathrm{L}^p(\mathbb{R}^d)$ for every $p \in [2,+\infty)$ and for every $p \in [2,+\infty]$ when $d = 1$. Consequently, for every n in $\mathbb{N}$, Z_n defines a $\mathrm{C}([0,T];\mathrm{H}^2(\mathbb{R}^d))$ random variable and therefore a $\mathrm{L}^{r(p)}\left(0,T;\mathrm{W}^{1,p}(\mathbb{R}^d)\right)$ random variable for the corresponding values of p.
Revisiting the proof of Proposition 3.1 in reference [9] and letting $2\sigma+2$ be replaced by any of the previous values of p besides $p = +\infty$ when $d = 1$, the necessary measurability issues to apply the Fubini's theorem are satisfied. Also, one gets the same

estimates and that there exists a constant $C(d,p)$ such that for every n and m in $\mathbb{N}$,

$$\mathbb{E}\left[\|Z_{n+m}(\omega)-Z_n(\omega)\|^r_{\mathrm{L}^{r(p)}(0,T;\mathrm{W}^{1,p}(\mathbb{R}^d))}\right]\le C(d,p)T^{\frac{r}{2}-1}\|\Phi_{n+m}-\Phi_n\|^r_{\mathcal{L}_2^{0,1}}.$$

The sequence $(Z_n)_{n\in\mathbb{N}}$ is thus a Cauchy sequence of $\mathrm{L}^r\left(\Omega;\mathrm{L}^r\left(0,T;\mathrm{W}^{1,p}(\mathbb{R}^d)\right)\right)$, which is a Banach space, and thus converges to $\tilde{Z}$. The previous calculation also gives that

$$\mathbb{E}\left[\|Z_n(\omega)-Z(\omega)\|^r_{\mathrm{L}^{r(p)}(0,T;\mathrm{L}^p(\mathbb{R}^d))}\right]\le C(d,p)T^{\frac{r}{2}-1}\|\Phi_n-\Phi\|^r_{\mathcal{L}_2^{0,1}}.$$

Therefore $\tilde{Z}=Z$, Z belongs to $\mathrm{L}^{r(p)}\left(0,T;\mathrm{W}^{1,p}(\mathbb{R}^d)\right)$ and it defines a measurable mapping as expected.
Remark that in $\mathcal{D}$, to simplify the notations, we did not write the cases $p=+\infty$ when $d=1$ or $p=\frac{2d}{d-2}$ when $d>2$. In fact, we are interested in results on the laws of the solutions of stochastic NLS and not really on the stochastic convolution. Also, the result of continuity in the next section shows that we necessarily loose on p in order to interpolate with $2<p<p'$ and have a nonzero exponent on the $\mathrm{L}^2(\mathbb{R}^d)-$norm. Therefore, even if it seems at first glance that we loose on the Sobolev's injections, it is not a restriction.

Step 2: The mapping Z is measurable with values in $\mathcal{D}$ with the Borel $\sigma-$field $\mathcal{B}^{\mathcal{D}}$. From step 1, given $x\in\mathcal{D}$, for every n in $\mathbb{N}^*$ such that $n>\frac{d-2}{4}$ the mapping $\omega\mapsto\|Z(\omega)-x\|_{X^{(n,p(n))}}$ from $(\Omega,\mathcal{F})$ into $(\mathbb{R}^+,\mathcal{B}(\mathbb{R}^+))$, where $\mathcal{B}(\mathbb{R}^+)$ stands for the Borel $\sigma-$field of $\mathbb{R}^+$, is measurable. Thus

$$\omega\mapsto d(Z(\omega),x)=\lim_{N\to+\infty}\sum_{n=1}^{N}\frac{1}{2^n}\left(\|Z(\omega)-x\|_{X^{(n,p(n))}}\vee 1\right)$$

is measurable. Consequently, for every r in $\mathbb{R}^+$, $\{\omega\in\Omega\ :\ d(Z(\omega),x)<r\}$ belongs to $\mathcal{F}$.
Remark that the law $\mu^{Z;\mathcal{D}}$ of Z on the metric space $\mathcal{D}$, which is a positive Borel measure, is therefore also regular and consequently it is a Radon measure.
The direct image of the Borel probability measure $\mu^{Z;\mathcal{D}}$ by the isomorphism defines the measure μ^Z on $\left(\mathcal{X}_\infty,\mathcal{B}^X\right)$.

Step 3 (Statements on the measures $\mu^{Z;(T,p)}$)**:** For (T,p) in the set of indices J, let $i_{(T,p)}$ denote the continuous injections from $X^{(T,p)}$ into $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))$ and $\mu^{Z;L}=(i_{(T,p)})_*\mu^{Z;(T,p)}$. The $\sigma-$field on $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))$ is the Borel $\sigma-$field. Let $h\in\mathrm{L}^2(0,T;L^2(\mathbb{R}^d))$, then

$$\left(h,i_{(T,p)}(Z)\right)_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))}=\int_0^T\sum_{i,j=1}^{+\infty}\int_0^t(e_j,S(t-s)\Phi e_i)_{\mathrm{L}^2(\mathbb{R}^d)}d\beta_i(s)(h(t),e_j)_{\mathrm{L}^2(\mathbb{R}^d)}$$

and from classical computation it is the almost sure limit of a sum of independent centered gaussian random variables, thus $\mu^{Z;L}$ is a centered gaussian measure.
Every linear continuous functional on $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))$ defines by restriction a linear continuous functional on $X^{(T,p)}$. Thus, $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))^*$ could be thought of as a subset of $\left(X^{(T,p)}\right)^*$. Since $i_{(T,p)}$ is a continuous injection, $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))^*$ is dense in $\left(X^{(T,p)}\right)^*$ for the weak* topology $\sigma\left(\left(X^{(T,p)}\right)^*,X^{(T,p)}\right)$. It means that, given

$x^* \in \left(X^{(T,p)}\right)^*$, there exists a sequence $(h_n)_{n\in\mathbb{N}}$ of elements of $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))$ such that for every $x \in X^{(T,p)}$,

$$\lim_{n\to+\infty} \left(h_n, i_{(T,p)}(x)\right)_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))} =< x^*, x >_{\left(X^{(T,p)}\right)^*, X^{(T,p)}} .$$

In other words, the random variable $< x^*, \cdot >_{\left(X^{(T,p)}\right)^*, X^{(T,p)}}$ is a pointwise limit of $\left(h_n, i_{(T,p)}(\cdot)\right)_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))}$ which are, from the above, centered gaussian random variables. As a consequence, $\mu^{Z;(T,p)}$ is a centered gaussian measure.
Recall that the RKHS $H_{\mu^{Z;L}}$ of $\mu^{Z;L}$ is $\mathrm{im}R^L$ where R^L is the mapping from $H^*_{\mu^{Z;L}} = \overline{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))^*}^{\mathrm{L}^2(\mu^{Z;L})}$ with the inner product derived from the one in $\mathrm{L}^2(\mu^{Z;L})$ into $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))$ defined for φ in $H^*_{\mu^{Z,L}}$ by

$$R^L(\varphi) = \int_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))} x\varphi(x)\mu^{Z;L}(dx).$$

The same is true for $H_{\mu^{Z;(T,p)}}$ replacing $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))$ by $X^{(T,p)}$ and $\mu^{Z;L}$ by $\mu^{Z;(T,p)}$.
Since $\mu^{Z;L}$ is the image of $\mu^{Z;(T,p)}$, taking $x^* \in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))^*$, we obtain that

$$\begin{aligned}
\|x^*\|_{\mathrm{L}^2(\mu^{Z;L})} &= \int_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))} < x^*, x >^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))^*, \mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))} \mu^{Z;L}(dx) \\
&= \int_{X^{(T,p)}} < x^*, x >^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))^*, \mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))} \mu^{Z;(T,p)}(dx) \\
&= \int_{X^{(T,p)}} < x^*, x >^2_{\left(X^{(T,p)}\right)^*, X^{(T,p)}} \mu^{Z;(T,p)}(dx) = \|x^*\|_{\mathrm{L}^2(\mu^{Z;(T,p)})}.
\end{aligned}$$

Therefore, from Lebesgue's dominated convergence theorem, we obtain that

$$\left(X^{(T,p)}\right)^* = \overline{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))^*}^{\sigma\left(\left(X^{(T,p)}\right)^*, X^{(T,p)}\right)} \subset \overline{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))^*}^{\mathrm{L}^2(\mu^{Z;(T,p)})}$$

where the last term is equal to $H^*_{\mu^{Z;L}}$. It follows that $H^*_{\mu^{Z;(T,p)}} \subset H^*_{\mu^{Z;L}}$.
The reverse inclusion follows from the fact that $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))^* \subset \left(X^{(T,p)}\right)^*$.
The conclusion follows from the quite standard fact that the RKHS of $\mu^{Z;L}$, which is a centered gaussian measure on a Hilbert space, is equal to $\mathrm{im}\mathcal{Q}^{\frac{1}{2}}$, with the norm of the image structure. $\mathcal{Q}$ denotes the covariance operator of the centered gaussian measure, it is given, see [8], for $h \in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))$, by

$$\mathcal{Q}h(v) = \int_0^T \int_0^{u\wedge v} IS(v-s)\Phi\Phi^* S(s-u)I^* h(u)dsdu.$$

Corollary B.5 of reference [8] finally gives that $\mathrm{im}\mathcal{L} = \mathrm{im}\mathcal{Q}^{\frac{1}{2}}$. □

2.5. **Continuity with respect to the perturbation.** Recall that the mild solution of stochastic NLS (1.3) could be written as a function of the perturbation. Let $v(x)$ denote the solution of

$$\left\{ \begin{array}{l} i\frac{\mathrm{d}}{\mathrm{d}t}v - (\Delta v + |v - ix|^{2\sigma}(v - ix)) = 0, \\ v(0) = u_0, \end{array} \right.$$

or equivalently a fixed point of the functional

$$\mathcal{F}_x(v)(t) = S(t)u_0 - i\lambda \int_0^t S(t-s)(|(v-ix)(s)|^{2\sigma}(v-ix)(s))ds,$$

where x is an element of $X^{(T,p)}$, p is such that $p \geq 2\sigma+2$ and (T,p) is an arbitrary pair in the set of indices J.
If u is such that $u = v(Z) - iZ$ where Z is the stochastic convolution, note that its regularity is given in the previous section, then u is a solution of (2.1) and of (1.2). Consequently, if $\mathcal{G}$ denotes the mapping that satisfies $\mathcal{G}(x) = v(x) - ix$ we obtain that $u = \mathcal{G}(Z)$.
The local existence follows from the fact that for $R > 0$ and $r > 0$ fixed, taking $\|x\|_{X^{(T,2\sigma+2)}} \leq R$ and $\|u_0\|_{\mathrm{H}^1(\mathbb{R}^d)} \leq r$, there exists a sufficiently small $T^*_{2\sigma+2}$ such that the closed ball centered at 0 of radius $2r$ is invariant and $\mathcal{F}_x$ is a contraction for the topology of $\mathrm{L}^\infty([0, T^*_{2\sigma+2}]; \mathrm{L}^2(\mathbb{R}^d)) \cap \mathrm{L}^r(0, T^*_{2\sigma+2}; \mathrm{L}^p(\mathbb{R}^d))$. The closed ball is complete for the weaker topology. The proof uses extensively the Strichartz' estimates, see [9] for a detailed proof. The same fixed point argument can be used for $\|x\|_{X^{(T,p)}} \leq R$ in a closed ball of radius $2r$ in $X^{(T^*_p,p)}$ for every T^*_p sufficiently small and $p \geq 2\sigma + 2$ such that $(T^*_p, p) \in J$. From (2.1), there exists a unique maximal solution $v(x)$ that belongs to $\mathcal{E}_\infty$.
It could be deduced from Proposition 3.5 of [9], that the mapping $\mathcal{G}$ from $\mathcal{X}_\infty$ into $\mathcal{E}_\infty$ is a continuous mapping from $\bigcap_{T\in\mathbb{R}^*_+} X^{(T,2\sigma+2)}$ with the projective limit topology into $\mathcal{E}(\mathrm{H}^1(\mathbb{R}^d))$. The result can be strengthen as follows.

Proposition 2.3. *The mapping $\mathcal{G}$ from $\mathcal{X}_\infty$ into $\mathcal{E}_\infty$ is continuous.*

Proof. Let $\tilde{x}$ be a function of $\mathcal{X}_\infty$ and $T < \mathcal{T}(\tilde{Z})$. Revisiting the proof of Proposition 3.5 of [9] and taking $\epsilon > 0$, $p' \geq 2\sigma + 2$, $2 < p < p'$, $R = 1 + \|\tilde{x}\|_{X^{(T,p')}}$ and $r = 1 + \|v(\tilde{x})\|_{\mathrm{C}([0,T];\mathrm{H}^1(\mathbb{R}^d))}$ there exists η satisfying

$$0 < \eta < \frac{\epsilon}{2(d+1)C(p,p')} \wedge 1$$

such for x in $\mathcal{X}_\infty$

$$\|x-\tilde{x}\|_{X^{(T,p')}} \leq \eta \Rightarrow \|v(x)-v(\tilde{x})\|_{\mathrm{C}([0,T];\mathrm{H}^1(\mathbb{R}^d))} \leq \left(\frac{\epsilon}{2(d+1)C(p,p')(4r)^\alpha}\right)^{\frac{1}{1-\alpha}} \wedge 1.$$

The constant α is the one that appears in the application of Hölder's inequality before (2.1). Consequently, since $v(x)$ and $v(\tilde{x})$ are functions of the closed ball centered at 0 and of radius $2r$ in $X^{(T,p)}$, the triangular inequality gives that

$$\|v(x) - v(\tilde{x})\|_{X^{(T,p')}} \leq 4r.$$

The application of both Hölder's inequality and the triangular inequality allow to conclude that

$$\forall x \in \mathcal{X}_\infty : \|x - \tilde{x}\|_{X^{(T,p')}} \leq \eta,\ \|\mathcal{G}(x) - \mathcal{G}(\tilde{x})\|_{X^{(T,p)}} \leq \epsilon$$

which, from the definition of the neighborhood basis of $\mathcal{E}_\infty$, gives the continuity. □

The following corollary is a consequence of the last statement of section 2.2.

Corollary 2.4. *In the focusing subcritical case or in the defocusing case, $\mathcal{G}$ is a continuous mapping from $\mathcal{X}_\infty$ into $\mathcal{X}_\infty$*

The continuity allows to define the law of the solutions of the stochastic NLS equations on $\mathcal{E}_\infty$ and in the cases of global existence in $\mathcal{X}_\infty$ as the direct image

$\mu^u = \mathcal{G}_*\mu^Z$, the same notation will be used in both cases.
Let consider the solutions of

$$i\mathrm{d}u_\epsilon - (\Delta u_\epsilon + \lambda|u_\epsilon|^{2\sigma}u_\epsilon)\mathrm{d}t = \sqrt{\epsilon}\mathrm{d}W, \tag{2.2}$$

where $\epsilon \geq 0$. The laws of the solutions u_ϵ in the corresponding trajectory spaces are denoted by μ^{u_ϵ}, or equivalently $\mathcal{G}_*\mu^{Z_\epsilon}$ where μ^{Z_ϵ} is the direct image of μ^Z under the transformation $x \mapsto \sqrt{\epsilon}x$ on $\mathcal{X}_\infty$. The continuity also gives that the family converges weakly to the Dirac mass on the deterministic solution u_d as ϵ converges to zero. Next section is devoted to the study of the convergence towards 0 of rare events or tail events of the law of the solution u_ϵ, namely large deviations. It allows to describe more precisely the convergence towards the deterministic measure.

3. Sample path large deviations

Theorem 3.1. *The family of probability measures $(\mu^{u_\epsilon})_{\epsilon\geq 0}$ on $\mathcal{E}_\infty$ satisfies a LDP of speed ϵ and good rate function*

$$I(u) = \frac{1}{2}\inf_{h\in\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d)):\mathbf{S}(h)=u}\left\{\|h\|^2_{\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d))}\right\},$$

where $\inf\emptyset = +\infty$ *and* $\mathbf{S}(h)$, *called the skeleton, is the unique mild solution of the following control problem:*

$$\begin{cases} i\frac{\mathrm{d}}{\mathrm{d}t}u = \Delta u + \lambda|u|^{2\sigma}u + \Phi h, \\ u(0) = u_0 \ \in \mathrm{H}^1(\mathbb{R}^d). \end{cases}$$

This is to say that for every Borel set A of $\mathcal{E}_\infty$,

$$-\inf_{u\in\overset{\circ}{A}} I(u) \leq \underline{\lim}_{\epsilon\to 0}\epsilon\log\mu^{u_\epsilon}(A) \leq \overline{\lim}_{\epsilon\to 0}\epsilon\log\mu^{u_\epsilon}(A) \leq -\inf_{u\in\overline{A}} I(u).$$

The same result holds in $\mathcal{X}_\infty$ for the family of laws of the solutions in the cases of global existence.

Proof. The general LDP for centered Gaussian measures on real Banach spaces, see [13], gives that for a given pair (T,p) in the set of indices J, the family $\left(p_{(T,p)*}\mu^{Z_\epsilon}\right)_{\epsilon\geq 0}$ satisfies a LDP on $X^{(T,p)}$ of speed ϵ and good rate function defined for $x \in X^{(T,p)}$ by,

$$I^{Z;(T,p)}(x) = \begin{cases} \frac{1}{2}\|x\|^2_{H_{\mu^{Z;(T,p)}}}, & \text{if } x \in H_{\mu^{Z;(T,p)}}, \\ +\infty, & \text{otherwise,} \end{cases}$$

which, using Proposition 2.2, is equal to

$$I^{Z;(T,p)}(x) = \frac{1}{2}\inf_{h\in\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d)):\mathcal{L}(h)=x}\left\{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))}\right\}.$$

Dawson-Gärtner's theorem, see [14], allows to deduce that the family $\left(\mu^{Z_\epsilon}\right)_{\epsilon\geq 0}$ satisfies the LDP with the good rate function defined for $x \in \mathcal{X}_\infty$ by

$$\begin{aligned} I^Z(x) &= \sup_{(T,p)\in J}\left\{I^{Z;(T,p)}(x)\right\} \\ &= \frac{1}{2}\inf_{h\in\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d)):\mathcal{L}(h)=x}\left\{\|h\|^2_{\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d))}\right\}. \end{aligned}$$

It has been shown in sections 2.2 and 2.5 that $\mathcal{G}$ is a continuous function from a Hausdorff topological space into another Hausdorff topological space. Consequently, both results follow from Varadhan's contraction principle along with the fact that if $\mathcal{G} \circ \mathcal{L}(h) = x$ then x is the unique mild solution of the control problem (i.e. $x = \mathbf{S}(h)$). □

Remark 3.2. The rate function is such that

$$I(u) = \frac{1}{2}\int_0^{\mathcal{T}(u)} \left\| \left(\Phi_{|\ker\Phi^\perp}\right)^{-1} \left(i\frac{\mathrm{d}}{\mathrm{d}t}u - \Delta u - \lambda|u|^{2\sigma}u\right)(s)\right\|_{\mathrm{L}^2(\mathbb{R}^d)} ds,$$

if $i\frac{\mathrm{d}}{\mathrm{d}t}u - \Delta u - \lambda|u|^{2\sigma}u \in \mathrm{im}\Phi$, and $I(u) = +\infty$ otherwise.

Remark 3.3. In the cases where blow-up may occur, the argument that will follow allows to prove the weaker result that, given an (T,p) in the set of indices J and

$$I^{(T,p)}(u) = \frac{1}{2}\inf_{h\in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d)):S(h)=u}\left\{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))}\right\},$$

then for every bounded Borel set A of $X^{(T,p)}$

$$-\inf_{u\in \overset{\circ}{A}} I^{(T,p)}(u) \leq \underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(u_\epsilon \in A\right) \leq \overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(u_\epsilon\in A\right) \leq -\inf_{u\in\overline{A}} I^{(T,p)}(u).$$

Indeed, if u_ϵ belongs to A, there exists a constant R such that $\|u_\epsilon\|_{X^{(T,p)}} \leq R$. Denoting by u_ϵ^R the solution of the following fixed point problem

$$u_\epsilon^R(t) = S(t)u_0 - i\lambda\int_0^t S(t-s)(|(u_\epsilon^R - i\sqrt{\epsilon}Z)(s)|^{2\sigma}(u_\epsilon^R - i\sqrt{\epsilon}Z)(s))\mathbb{1}_{\|u_\epsilon^R\|_{X^{(s,p)}}\leq R}ds,$$

the arguments used previously allow to show that $\sqrt{\epsilon}Z \to u_\epsilon^R$ is a continuous mapping from every $X^{(T,p')}$ into $X^{(T,p)}$ for $p' > p$. The result on the laws of u_ϵ^R follows from Varadhan's contraction principle replacing $S(h)$ by $S^R(h)$ with the truncation in front of the nonlinearity. Finally, the statement follows from the fact that $\|u_\epsilon\|_{X^{(T,p)}} \leq R$ implies that $u_\epsilon^R = u_\epsilon$ and that

$$\inf_{h\in\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d)):S^R(h)\in\overline{A}}\{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))}\} = \inf_{h\in\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d)):S(h)\in\overline{A}}\{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}^d))}\}.$$

Remark that writing $\frac{\partial}{\partial t}h$ instead of h in the optimal control problem leads to a rate function consisting in the minimisation of $\frac{1}{2}\|h\|^2_{H_0^1(0,+\infty;\mathrm{L}^2(\mathbb{R}^d))}$. This space is somehow the equivalent of the Cameron-Martin space for the Brownian motion. Specifying only the law μ of $W(1)$ on $\mathrm{H}^1(\mathbb{R}^d)$ and dropping Φ in the control problem would lead to a rate function consisting in the minimisation of $\frac{1}{2}\|h\|^2_{H_0^1(0,+\infty;H_\mu)}$, where H_μ stands for the RKHS of μ.

The formalism of a LDP stated in the intersection space with a projective limit topology allows, for example, to deduce by contraction, when there is no blow-up in finite time, a variety of sample path LDP on every $X^{(T,p)}$. The rate function could be interpreted as the minimal energy to implement control.

LDP for the family of laws of $u_\epsilon(T)$, for a fixed T, could be deduced by contraction in the cases of global existence. The rate function is then the minimal energy needed to transfer u_0 to x from 0 to T. An application of this type will be given in section 6.

Next section gives a characterization of the support of the law of the solution in our setting. Section 5 is devoted to some consequences of these results on the blow-up time. Finally, in section 6, applications in nonlinear optics are given.

4. Remark on the support of the law of the solution

Theorem 4.1 (The support theorem). *The support of the law of the solution is characterized by*

$$supp\ \mu^u = \overline{\text{im}\mathbf{S}}^{\mathcal{E}_\infty}$$

and in the cases of global existence by

$$supp\ \mu^u = \overline{\text{im}\mathbf{S}}^{\mathcal{X}_\infty}$$

Proof. **Step 1:** From Proposition 2.3, given (T,p) in the set of indices J, $\mu^{Z;(T,p)}$ is a gaussian measure on a Banach space and its RKHS is $\text{im}\mathcal{L}$. Consequently, see [2] Theorem (IX,2;1), its support is $\overline{\text{im}\mathcal{L}}^{X^{(T,p)}}$. Also, from the definition of the image measure we have that

$$\mu^Z\left(p_{(T,p)}^{-1}\left(\overline{\text{im}\mathcal{L}}^{X^{(T,p)}}\right)\right) = \mu^{Z;(T,p)}\left(\overline{\text{im}\mathcal{L}}^{X^{(T,p)}}\right) = 1.$$

As a consequence the first inclusion follows

$$supp\ \mu^Z \subset \bigcap_{(T,p)} p_{(T,p)}^{-1}\left(\overline{\text{im}\mathcal{L}}^{X^{(T,p)}}\right) = \overline{\text{im}\mathcal{L}}^{\mathcal{X}_\infty}.$$

It then suffices to show that $\text{im}\mathcal{L} \subset supp\ \mu^Z$. Suppose that $x \notin supp\ \mu^Z$, then there exists a neighborhood V of x in $\mathcal{X}_\infty$, satisfying $V = \bigcap_{i=1}^n V^{(T_i,p_i)}$ where $V^{(T_i,p_i)}$ is a neighborhood of x in $X^{(T_i,p_i)}$, n is a finite integer and (T_i,p_i) a finite sequence of elements of J, such that $\mu^Z(V) = 0$. It can be shown that $\bigcap_{i=1}^n X^{(T_i,p_i)}$ is still a separable Banach space. It is such that $\mathcal{D}$ is continuously embedded into it, and such that the Borel direct image probability measure is a Gaussian measure of RKHS $\text{im}\mathcal{L}$. The support of this measure is then the adherence of $\text{im}\mathcal{L}$ for the topology defined by the maximum of the norms on each factor. Thus, $V \cap \text{im}\mathcal{L} = \emptyset$ and $x \notin \text{im}\mathcal{L}$.

Step 2: We conclude using the continuity of $\mathcal{G}$.
Indeed since $\mathcal{G}(\text{im}\mathcal{L}) \subset \overline{\mathcal{G}(\text{im}\mathcal{L})}^{\mathcal{E}_\infty}$, $\text{im}\mathcal{L} \subset \mathcal{G}^{-1}\left(\overline{\mathcal{G}(\text{im}\mathcal{L})}^{\mathcal{E}_\infty}\right)$. Since $\mathcal{G}$ is continuous, the right side is a closed set of $\mathcal{X}_\infty$ and from step 1,

$$supp\ \mu^Z \subset \mathcal{G}^{-1}\left(\overline{\text{im}\left(\mathcal{G}\circ\mathcal{L}\right)}^{\mathcal{E}_\infty}\right),$$

and

$$\mu^Z\left(\mathcal{G}^{-1}\left(\overline{\text{im}\mathbf{S}}^{\mathcal{E}_\infty}\right)\right) = 1,$$

thus

$$supp\ \mu^u \subset \overline{\text{im}\mathbf{S}}^{\mathcal{E}_\infty}.$$

Suppose that $x \notin supp\ \mu^u$, there exists a neighborhood V of x in $\mathcal{E}_\infty$ such that $\mu^u(V) = \mu^Z\left(\mathcal{G}^{-1}(V)\right) = 0$, consequently $\mathcal{G}^{-1}(V) \bigcap \text{im}\mathcal{L} = \emptyset$ and $x \notin \text{im}\mathbf{S}$. This gives reverse inclusion.
The same arguments hold replacing $\mathcal{E}_\infty$ by $\mathcal{X}_\infty$. □

Remark that the result of step 2 is general and gives that the support of the direct images μ^E of the law μ^u by any continuous mapping f from either $\mathcal{E}_\infty$ or $\mathcal{X}_\infty$ into a topological vector space E is $\overline{\text{im}\left(f\circ\mathbf{S}\right)}^E$. For example, in the cases of global existence, given a positive T, the support of the law in $\mathrm{H}^1(\mathbb{R}^d)$ of $u(T)$ is $\overline{\text{im}\mathbf{S}(T)}^{\mathrm{H}^1(\mathbb{R}^d)}$.

5. Applications to the blow-up time

In this section the equation with a focusing nonlinearity, i.e. $\lambda = 1$, is considered. In this case, it is known that some solutions of the deterministic equation blow up in finite time. It has been proved in section 2.2 that $\mathcal{T}$ is a measurable mapping from $\mathcal{E}_\infty$ to $[0, +\infty]$, both spaces are equipped with their Borel $\sigma-$fields. Incidentally, $\mathcal{T}(u)$ is a $\mathcal{F}_t-$stopping time. Also, if B is a Borel set of $[0, +\infty]$,

$$\mathbb{P}\left(\mathcal{T}(u) \in B\right) = \mu^u \left(\mathcal{T}^{-1}(B)\right).$$

The support theorem allows to determine whether an open or a closed set of the form $\mathcal{T}^{-1}(B)$ is such that $\mu^u \left(\mathcal{T}^{-1}(B)\right) > 0$ or $\mu^u \left(\mathcal{T}^{-1}(B)\right) < 1$ respectively. An application of this fact is given in Proposition 5.1. For a Borel set B such that the interior of $\mathcal{T}^{-1}(B) \cap \overline{\mathrm{im}\mathbf{S}}^{\mathcal{E}_\infty}$ is nonempty, $\mathbb{P}(\mathcal{T}(u) \in B) > 0$ holds.

Also, $\mathcal{T}$ is not continuous and Varadhan's contraction principle does not allow to obtain a LDP for the law of the blow-up time. Nonetheless, the LDP for the family $(\mu^{u_\epsilon})_{\epsilon>0}$ gives the interesting result that

$$-\inf_{u \in Int(\mathcal{T}^{-1}(B))} I(u) \leq \underline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\mathcal{T}(u_\epsilon) \in B\right)$$

and that

$$\overline{\lim}_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(\mathcal{T}(u_\epsilon) \in B\right) \leq -\inf_{u \in \overline{\mathcal{T}^{-1}(B)}} I(u),$$

where $Int\left(\mathcal{T}^{-1}(B)\right)$ stands for the interior set of $\mathcal{T}^{-1}(B)$. Remark also that the interior or the adherence of sets in $\mathcal{E}_\infty$ are not really tractable. In that respect, the semicontinuity of $\mathcal{T}$ makes the sets $(T, +\infty]$ and $[0, T]$ particularly interesting.

5.1. **Probability of blow-up after time T.**

Proposition 5.1. *If* $u_0 \in \mathrm{H}^3(\mathbb{R}^d)$ *and* $\ker\Phi^* = \{0\}$ *then for every positive* T,

$$\mathbb{P}(\mathcal{T}(u) > T) > 0.$$

Proof. Since $\mathcal{T}$ is lower semicontinuous, $\mathcal{T}^{-1}((T, +\infty])$ is an open set.
Consider $H = -\Delta u_0 - |u_0|^{2\sigma} u_0$ which satisfies $\mathcal{G} \circ \Lambda(H) = u_0$, where Λ has been defined in section 2.1, then $\mathcal{T}(\mathbf{S}(H)) = +\infty$. Also, Φ defines an operator from $\mathrm{L}^2{}_{loc}(0, +\infty; \mathrm{L}^2(\mathbb{R}^d))$ into $\mathrm{L}^2{}_{loc}(0, +\infty; \mathrm{H}^1(\mathbb{R}^d))$ and it can be shown, since $\ker\Phi^* = \{0\}$, that its range is dense. Consequently, there exists a sequence $(h_n)_{n\in\mathbb{N}}$ of $\mathrm{L}^2{}_{loc}(0, +\infty; \mathrm{L}^2(\mathbb{R}^d))$ functions such that $(\Phi(h_n))_{n\in\mathbb{N}}$ converges to H in $\mathrm{L}^2{}_{loc}(0, +\infty; \mathrm{H}^1(\mathbb{R}^d))$.
Using the semicontinuity of $\mathcal{T}$, the continuity of $\mathcal{G}$, the fact that $\mathbf{S} = \mathcal{G} \circ \Lambda \circ \Phi$, the following Lemma and the fact that $\mathrm{L}^2{}_{loc}(0, +\infty; \mathrm{H}^1(\mathbb{R}^d))$ is continuously embedded in $\mathrm{L}^1_{loc}(0, +\infty; \mathrm{H}^1(\mathbb{R}^d))$, $\underline{\lim}_{n\to\infty} \mathcal{T}(\mathbf{S}(h_n)) \geq +\infty$, i.e. $\lim_{n\to\infty} \mathcal{T}(\mathbf{S}(h_n)) = +\infty$, follows. Therefore $\mathcal{T}(\mathbf{S}(h_n)) > T$ for n large enough and $\mathcal{T}^{-1}((T, +\infty]) \cap (\mathrm{im}\mathbf{S})$ is nonempty.
The conclusion follows then from the support theorem. □

As a corollary, taking the complementary of $\mathcal{T}^{-1}((T, +\infty])$, $\mathbb{P}(\mathcal{T}(u) \leq T) < 1$ follows. This is related to the results of [10] where it is proved that for every positive T, $\mathbb{P}(\mathcal{T}(u) < T) > 0$ and to the graphs in section 4 of [12].

Lemma 5.2. *The operator* Λ *from* $\mathrm{L}^1_{loc}(0, +\infty; \mathrm{H}^1(\mathbb{R}^d))$ *into* $\mathcal{X}_\infty$ *of ii/ of the Strichartz estimates is continuous.*

Proof. The result follows from $ii/$ of the Strichartz estimates, the fact that the partial derivatives with respect to one space variable commutes with both the integral and the group and the definition of the projective limit topology. □

The following result holds when the intensity of the noise converges to zero.

Proposition 5.3. *If $u_0 \in \mathrm{H}^3(\mathbb{R}^d)$, $\ker\Phi^* = \{0\}$ and $T \geq \mathcal{T}(u_d)$, where u_d is the solution of the deterministic NLS equation with initial datum u_0, there exists a positive constant c such that*

$$\underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}(u_\epsilon) > T\right) \geq -c.$$

Proof. The result follows from

$$-\frac{1}{2}\inf_{h\in \mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d)):\mathcal{T}(\mathbf{S}(h))>T}\left\{\|h\|^2_{\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d))}\right\} \leq \underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}(u_\epsilon) > T\right)$$

and the fact that, from the arguments of the proof of Proposition 5.1, for every T such that $T \geq \mathcal{T}(u_d)$ the set $\{h \in \mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d)) : \mathcal{T}(\mathbf{S}(h) > T\}$ is nonempty. □

In the following we will denote by $L^{(T,+\infty)}$ the infimum in the left hand side of the inequality of the above proof.

Remark 5.4. The assumption that $u_0 \in \mathrm{H}^3(\mathbb{R}^d)$ could be dropped using similar arguments as in Proposition 3.3 of [10].

Remark that the LDP does not give interesting information on the upper bound even if the bounds have been sharpened using the rather strong projective limit topology. It is zero since $h = 0$ belongs to $\overline{\mathcal{T}^{-1}((T,+\infty])}$ as for every $T > 0$, $\overline{\mathcal{T}^{-1}((T,+\infty])} = \mathcal{E}_\infty$. Indeed, if a function f of $\mathcal{E}_\infty$ is given and blows up at a particular time $\mathcal{T}(f)$ such that $T > \mathcal{T}(f)$, it is possible to build a sequence $(f_n)_{n\in\mathbb{N}}$ of functions of $\mathcal{E}_\infty$ equal to f on $\left[0, \mathcal{T}(f) - \frac{1}{n}\right]$ and such that $\mathcal{T}(f_n) > T$. The same problem will appear in the next section where the LDP gives a lower bound equal to $-\infty$. Indeed, $Int\left(\mathcal{T}^{-1}([0,T])\right)$ is the complementary of the above and thus an empty set. To overcome this problem the approximate blow-up time is introduced. Remark also that it is possible that $L^{(T,+\infty)} = 0$.

Also, the case $T < \mathcal{T}(u_d)$ has not been treated. Indeed, the associated event is not a large deviation event and the LDP only gives that

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(\mathcal{T}(u_\epsilon) > T\right) = 0.$$

5.2. **Probability of blow-up before time T.** In that case we obtain

$$-\infty \leq \underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}(u_\epsilon) \leq T\right) \leq \overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}(u_\epsilon) \leq T\right) \leq -U^{[0,T]}$$

where $U^{[0,T]} = \frac{1}{2}\inf_{h\in \mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d)):\mathcal{T}(\mathbf{S}(h))\leq T}\left\{\|h\|^2_{\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d))}\right\}$.

Proposition 5.5. *If $T < \mathcal{T}(u_d)$,*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}(u_\epsilon) \leq T\right) \leq -U^{[0,T]} < 0.$$

Moreover, if $u_0 \in \mathrm{H}^3(\mathbb{R}^d)$ and if u_0, Δu_0 and $|u_0|^{2\sigma}u_0$ belong to $\mathrm{im}\Phi$ then

$$-\infty < \underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}\left(\mathcal{T}(u_\epsilon) \leq T\right).$$

Proof. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of $\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d))$ functions converging to zero. It follows from Lemma 5.2 and the fact that $\mathrm{L}^2(0,+\infty;\mathrm{H}^1(\mathbb{R}^d))$ is continuously embedded into $\mathrm{L}^1_{loc}(0,+\infty;\mathrm{H}^1(\mathbb{R}^d))$ that $\mathbf{S}=\mathcal{G}\circ\Lambda\circ\Phi$ is continuous from $\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d))$ into $\mathcal{E}_\infty$.
Also, from the semicontinuity of $\mathcal{T}$, $\underline{\lim}_{\epsilon\to 0}\mathcal{T}(\mathbf{S}(h_n))\geq\mathcal{T}(u_d)$ and the first point follows.
The $\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d))$ control

$$\begin{aligned}H^{\mathcal{E}}(t)=&\frac{2}{T-2t}1\!\!1_{t\leq\frac{T}{2}}\left[-\frac{2i}{T-2t}\left(\Phi_{|\ker\Phi^\perp}\right)^{-1}u_0-\left(\Phi_{|\ker\Phi^\perp}\right)^{-1}(\Delta u_0)\right.\\&\left.-\left(\frac{2}{T-2t}\right)^2\left(\Phi_{|\ker\Phi^\perp}\right)^{-1}\left(|u_0|^{2\sigma}u_0\right)\right]\end{aligned}$$

is such that $\mathbf{S}(H^{\mathcal{E}})=\frac{2}{T-2t}u_0$ which blows up before T. This proves the second point. □

When $T\geq\mathcal{T}(u_d)$, the probability is not supposed to tend to zero. Also, as $h=0$ is a solution, the upper bound is zero and none of the bounds are interesting.

5.3. **Bounds for the approximate blow-up time.** To overcome the limitation that $\overline{\mathcal{T}^{-1}((T,+\infty))}=\mathcal{E}_\infty$, which does not allow to have two interesting bounds simultaneously, we introduce for every positive R the mappings $\mathcal{T}_R$ defined for $f\in\mathcal{E}_\infty$ by

$$\mathcal{T}_R(f)=\inf\{t\in[0,+\infty):\ \|f(t)\|_{\mathrm{H}^1(\mathbb{R}^d)}\geq R\}.$$

It corresponds to the approximation of the blow-up time used in [12]. We obtain the following bounds.

Proposition 5.6. *When $T\geq\mathcal{T}_R(u_d)$, the following inequality holds*

$$-c<-L_R^{(T,+\infty)}\leq\underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathcal{T}_R(u_\epsilon)>T\right)$$

and

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathcal{T}_R(u_\epsilon)>T\right)\leq-\sup_{\alpha>0}L_{R+\alpha}^{(T,+\infty)}.$$

Also, when $T<\mathcal{T}_R(u_d)$, we have that

$$-\inf_{\alpha>0}U_{R+\alpha}^{[0,T]}\leq\underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathcal{T}_R(u_\epsilon)\leq T\right)$$

and

$$\overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}\left(\mathcal{T}_R(u_\epsilon)\leq T\right)\leq-U_R^{[0,T]}<0.$$

In the above, c is nonnegative and the numbers $L_R^{(T,+\infty)}$ and $U_R^{[0,T]}$ are defined as $L^{(T,+\infty)}$ and $U^{[0,T]}$ replacing $\mathcal{T}$ by $\mathcal{T}_R$.

Proof. The result follows from the facts that $\mathcal{T}_R$, which is not continuous, is lower semicontinuous, that for every positive α, $\overline{\mathcal{T}_R^{-1}((T,+\infty))}\subset\mathcal{T}_{R+\alpha}^{-1}((T,+\infty))$, thus $\mathcal{T}_{R+\alpha}^{-1}([0,T])\subset Int\left(\mathcal{T}_R^{-1}([0,T])\right)$ and from the arguments used in the proofs of the last two propositions. □

We also obtain the following estimates of other large deviation events.

Corollary 5.7. *If $S,T < \mathcal{T}_R(u_d)$, for every positive c, there exists a positive ϵ_0 such that if $\epsilon \le \epsilon_0$,*

$$L^{S,T}_{<,R,\epsilon,c} \le \mathbb{P}\left(S < \mathcal{T}_R(u_\epsilon) \le T\right) \le U^{S,T}_{<,R,\epsilon,c}$$

where

$$L^{S,T}_{<,R,\epsilon,c} = \exp\left(-\frac{\inf_{\alpha>0} U^{[0,T]}_{R+\alpha} + c}{\epsilon}\right)\left(1-\exp\left(-\frac{U^{[0,S]}_{R} - \inf_{\alpha>0} U^{[0,T]}_{R+\alpha}}{\epsilon}\right)\right)$$

and

$$U^{S,T}_{<,R,\epsilon,c} = \exp\left(-\frac{U^{[0,T]}_{R} - c}{\epsilon}\right)\left(1-\exp\left(-\frac{\inf_{\alpha>0} U^{[0,S]}_{R+\alpha} - U^{[0,T]}_{R}}{\epsilon}\right)\right).$$

Also, if $S,T > \mathcal{T}_R(u_d)$, for every positive c, there exists a positive ϵ_0 such that if $\epsilon \le \epsilon_0$,

$$L^{S,T}_{>,R,\epsilon,c} \le \mathbb{P}\left(S < \mathcal{T}_R(u_\epsilon) \le T\right) \le U^{S,T}_{>,R,\epsilon,c}$$

where

$$L^{S,T}_{>,R,\epsilon,c} = \exp\left(-\frac{L^{(S,+\infty)}_{R} + c}{\epsilon}\right)\left(1-\exp\left(-\frac{\sup_{\alpha>0} L^{(T,+\infty)}_{R+\alpha} - L^{(T,+\infty)}_{R}}{\epsilon}\right)\right)$$

and

$$U^{S,T}_{>,R,\epsilon,c} = \exp\left(-\frac{\sup_{\alpha>0} L^{(S,+\infty)}_{R+\alpha} - c}{\epsilon}\right)\left(1-\exp\left(-\frac{L^{(T,+\infty)}_{R} - \sup_{\alpha>0} L^{(S,+\infty)}_{R+\alpha}}{\epsilon}\right)\right).$$

Proof. When $S,T < \mathcal{T}_R(u_d)$, the result follows from the inequalities and from the fact that

$$\begin{aligned}\mathbb{P}\left(S < \mathcal{T}_R(u_\epsilon) \le T\right) &= \mathbb{P}\left(\{\mathcal{T}_R(u_\epsilon) \le T\} \setminus \{\mathcal{T}_R(u_\epsilon) \le S\}\right)\\ &= \mathbb{P}\left(\mathcal{T}_R(u_\epsilon) \le T\right)\left(1 - \frac{\mathbb{P}\left(\mathcal{T}_R(u_\epsilon) \le S\right)}{\mathbb{P}\left(\mathcal{T}_R(u_\epsilon) \le T\right)}\right).\end{aligned}$$

When $S,T > \mathcal{T}_R(u_d)$, we use

$$\begin{aligned}\mathbb{P}\left(S < \mathcal{T}_R(u_\epsilon) \le T\right) &= \mathbb{P}\left(\{\mathcal{T}_R(u_\epsilon) > S\} \setminus \{\mathcal{T}_R(u_\epsilon) > T\}\right)\\ &= \mathbb{P}\left(\mathcal{T}_R(u_\epsilon) > S\right)\left(1 - \frac{\mathbb{P}\left(\mathcal{T}_R(u_\epsilon) > T\right)}{\mathbb{P}\left(\mathcal{T}_R(u_\epsilon) > S\right)}\right).\end{aligned}$$

□

6. APPLICATIONS TO NONLINEAR OPTICS

The NLS equation when $d = 1$, $\sigma = 1$ and $\lambda = 1$ is called the noisy cubic focusing nonlinear Schrödinger equation. It is a model used in nonlinear optics. Recall that for the above values of the parameters the solutions are global. The variable t stands for the one dimensional space coordinate and x for the time. The deterministic equation is such that there exists a particular class of solutions, which are localized in space (here time), that propagate at a finite constant velocity and keep the same shape. These solutions are called solitons or solitary waves. The functions

$$\Psi_\eta(t,x) = \sqrt{2}\eta \exp\left(-i\eta^2 t\right)\operatorname{sech}(\eta x),\ \ \eta > 0,$$

form a family of solitons. They are used in optical fibers as information carriers to transmit the datum 0 or 1 at high bit rates over long distances. The noise stands for the noise produced by in-line amplifiers.

Let u_ϵ denote the solution with $u_0(\cdot) = \Psi_1(0,\cdot)$ as initial datum and ϵ as noise intensity like in section 3 and u_ϵ^n denote the solution with null initial datum and the same noise intensity. The square of the momentum of u_0 is 4.

At a particular coordinate T of the fiber, when a window $[-l,l]$ is given, the square of the $\mathrm{L}^2(-l,l)-$norm, or measured square of the momentum, is recorded. It is close to the momentum in the deterministic case for sufficiently high l since the wave is localized. A decision criterium is to accept that we have 1 if the measured square of the momentum is above a certain threshold and 0 otherwise. We set a threshold of the form $4(1-\gamma)$, where γ is a real number in $[0,1]$.

As the soliton is progressively distorted by the noise, it is possible either to wrongly decide that the source has emitted a 1, or to wrongly discard a 1. The two error probabilities consist of

$$\mathbb{P}_\epsilon^{|0} = \mathbb{P}\left(\int_{-l}^{l} |u_\epsilon^n(T,x)|^2 dx \geq 4(1-\gamma)\right)$$

and

$$\mathbb{P}_\epsilon^{|1} = \mathbb{P}\left(\int_{-l}^{l} |u_\epsilon(T,x)|^2 dx < 4(1-\gamma)\right).$$

In the following we make the assumption that $\ker\Phi^* = \{0\}$. Indeed, from the arguments used in the proof of Proposition 5.3, it is needed for controllability issues to guaranty that the infima are not taken over empty sets. Also T is fixed, $\gamma_0 \in \left(0,\frac{1}{2}\right)$ is fixed and the size l of the window is such that

$$\int_{-l}^{l} |u_d(T,x)|^2 dx \wedge \int_{-l}^{l} |\Psi_1(0,x)|^2 dx > 4\left(1-\frac{\gamma_0}{2}\right).$$

Proposition 6.1. *For every γ in $[\gamma_0, 1-\gamma_0]$ besides an at most countable set of points, the following equivalents for the probabilities of error hold*

$$\log \mathbb{P}_\epsilon^{|0} \sim_{\epsilon\to 0} -\frac{1}{2\epsilon} \inf_{h\in \mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R})):\int_{-l}^{l} |\tilde{\mathbf{S}}(h)(T,x)|^2 dx \geq 4(1-\gamma)} \left\{\|h\|^2_{\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}))}\right\}$$

$$\log \mathbb{P}_\epsilon^{|1} \sim_{\epsilon\to 0} -\frac{1}{2\epsilon} \inf_{h\in \mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R})):\int_{-l}^{l} |\mathbf{S}(h)(T,x)|^2 dx < 4(1-\gamma)} \left\{\|h\|^2_{\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}))}\right\}$$

where $\tilde{\mathbf{S}}(h)$ is the skeleton associated to the same control problem as $\mathbf{S}(h)$ but with null initial datum. Both infima are positive numbers.

Proof. The mapping φ from $\mathcal{X}_\infty$ into $\mathbb{R}^+$ such that $\varphi(f) = \int_{-l}^{l} |f(x)|^2 dx$ is continuous. Therefore, the direct image measures $\left(\varphi_* \mu^{u_\epsilon}\right)_{\epsilon\geq 0}$ and $\left(\varphi_* \mu^{u_\epsilon^n}\right)_{\epsilon\geq 0}$ satisfy LDP of speed ϵ and good rate functions respectively

$$I^T(y) = \frac{1}{2} \inf_{h\in \mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R})):\int_{-l}^{l} |\mathbf{S}(h)(T,x)|^2 dx = y} \left\{\|h\|^2_{\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}))}\right\}$$

and J^T where $\mathbf{S}$ is replaced by $\tilde{\mathbf{S}}$. Consequently,

$$\forall i \in \{0,1\},\ -L^i(\gamma) \le \underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}_\epsilon^i \le \overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}_\epsilon^i \le -U^i(\gamma)$$

where

$$\begin{aligned} L^0(\gamma) &= \inf_{y\in(4(1-\gamma),+\infty)} J^T(y), \quad U^0(\gamma) = \inf_{y\in[4(1-\gamma),+\infty)} J^T(y),\\ L^1(\gamma) &= \inf_{y\in[0,4(1-\gamma))} I^T(y), \qquad U^1(\gamma) = \inf_{y\in[0,4(1-\gamma)]} I^T(y). \end{aligned}$$

For every $\delta > 0$, $U^0(\gamma) \le L^0(\gamma) \le U^0(\gamma-\delta)$ and $U^1(\gamma) \le L^1(\gamma) \le U^1(\gamma+\delta)$ hold. The function $\gamma \mapsto U^0(\gamma)$ is positive and decreasing. Also, as $\ker\Phi^* = \{0\}$, there exists a sequence $(h_n^0)_{n\in\mathbb{N}}$ of functions of $\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}))$ that converges to

$$H^0(t) = i\frac{\mathrm{d}}{\mathrm{d}t}u^0 - \Delta u^0 - \lambda |u^0|^{2\sigma}u^0$$

where

$$u^0(t) = \mathbb{1}_{t\le T}\frac{t}{T}\Psi_1(0,\cdot)$$

and by the continuity proved in section 5.1 $\left(\varphi\circ\mathbf{S}(h_n^0)\right)_{n\in\mathbb{N}}$ converges to $\varphi\circ\mathbf{S}(H^0)$ which is such that $\varphi\circ\mathbf{S}(H^0) > 4\left(1-\frac{\gamma_0}{2}\right) > 4(1-\gamma_0)$. Consequently, h_n^0 belongs to the minimizing set for n large enough. Thus, $U^0(\gamma_0) < +\infty$ follows. Consequently, the function $\gamma\mapsto U^0(\gamma)$ possesses an at most countable set of points of discontinuity. Similarly, the function $\gamma\mapsto U^1(\gamma)$ is a bounded increasing function. Also, if $(h_n^1)_{n\in\mathbb{N}}$ and $H^1(t)$ are defined as previously replacing $u^0(t)$ by

$$u^1(t) = \mathbb{1}_{t\le T}\left(1-\left(1-\sqrt{\frac{\gamma_0}{2}}\right)\frac{t}{T}\right)\Psi_1(0,\cdot),$$

the sequence $(\varphi\circ\mathbf{S}(h_n^1))_{n\in\mathbb{N}}$ converges to $\varphi\circ\mathbf{S}(H^1) \le 2\gamma_0 = 4\left(1-\left(1-\frac{\gamma_0}{2}\right)\right)$. Thus, for n large enough h_n^1 belongs to the minimizing set. Consequently, the function $\gamma\mapsto U^1(\gamma)$ has an at most countable set of points of discontinuity. Thus, for a well chosen γ, letting δ converge to zero, we obtain for $i\in\{0,1\}$ that $L^i(\gamma) = U^i(\gamma)$ and the equivalents follow.
From the arguments used in the proof of Proposition 5.5, $\tilde{\mathbf{S}}$ is a continuous mapping from $\mathrm{L}^2(0,+\infty;\mathrm{H}^1(\mathbb{R}))$ into $\mathcal{X}_\infty$. Since φ is continuous, if $(H_n)_{n\in\mathbb{N}}$ is a sequence of functions converging to zero in $\mathrm{L}^2(0,+\infty;\mathrm{H}^1(\mathbb{R}))$ then $\left(\varphi\circ\tilde{\mathbf{S}}(H_n)\right)_{n\in\mathbb{N}}$ converges to $\varphi\circ\tilde{\mathbf{S}}(0) = 0$. Proposition 5.5 also gives that $(\varphi\circ\mathbf{S}(H_n))_{n\in\mathbb{N}}$ converges to $\varphi\circ\mathbf{S}(0)$ which satisfies $\varphi\circ\mathbf{S}(0) > 4\left(1-\frac{\gamma_0}{2}\right)$. The conclusion follows. $\square$

In reference [16] the authors explain that, for the second error probability, two processes are mainly responsible for the deviations of the measured square of the momentum from its expected value: the fluctuation of the soliton power

$$\frac{M(u_\epsilon(T))^2}{4}$$

and a shift of the soliton position, also called center of mass, timing jitter or fluctuation in timing since time and space variables have been exchanged, characterized by

$$\frac{\int_{-\infty}^{+\infty} x|u_\epsilon(T,x)|^2dx}{M(u_\epsilon(T))^2}.$$

Recall that M stands for the momentum or $\mathrm{L}^2(\mathbb{R})-$norm. The authors give an asymptotic expression of the probability density function of the joint law of the two above random variables. In the case of the first error probability, they explain that the optimal way to create a large signal is to grow a soliton, they thus give the marginal probability density function of the square of the momentum.

In the two following sections we concentrate on the square of the momentum, we take $l = +\infty$ as if the window were not bounded. Somehow, if we forget the coefficient, we concentrate on the tails of the marginal law of the soliton power when ϵ converges to zero. We recall, it has been pointed out in the introduction, that the momentum is no longer preserved in the stochastic case and is such that its expected value increases. We then study the tails of the law of the shift of the soliton position when ϵ converges to zero. Remark that we drop the renormalization in the shift of the soliton position so as to obtain a probability measure in the integral and write

$$Y_\epsilon = \int_{-\infty}^{+\infty} x|u_\epsilon(T,x)|^2 dx.$$

We finally present a result for the general case, with no limitation on σ and d, where blow-up may occur.

6.1. **Upper bounds.** The norm of the linear continuous operator Φ of $\mathrm{L}^2(\mathbb{R})$ is thereafter denoted by $\|\Phi\|_c$.

Proposition 6.2. *For every positive T, γ in $[0,1]$, and every operator Φ in $\mathcal{L}_2(\mathrm{L}^2(\mathbb{R}),\mathrm{H}^1(\mathbb{R}))$, the inequalities*

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}_\epsilon^{|0} \leq -\frac{1-\gamma}{2T\|\Phi\|_c^2}$$

and

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}_\epsilon^{|1} \leq -\frac{1+\gamma}{T\|\Phi\|_c^2}\left(\sqrt{1+\left(\frac{\gamma}{1+\gamma}\right)^2}-1\right).$$

hold.

Proof. Multiplying by $-i\overline{u}$ the equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}u - \Delta u - \lambda|u|^{2\sigma}u = \Phi h,$$

integrating over time and taking the real part gives that

$$\|u(T)\|^2_{\mathrm{L}^2(\mathbb{R})} - \|u_0\|^2_{\mathrm{L}^2(\mathbb{R})} = 2\Re\left(-i\int_0^T\int_{\mathbb{R}}\Phi h\overline{u}\; dxdt\right).$$

First bound: The boundary conditions $\|u(T)\|^2_{\mathrm{L}^2(\mathbb{R})} \geq 4(1-\gamma)$ and $u_0 = 0$ along with Cauchy Schwarz inequality imply both that

$$4(1-\gamma) \leq 2\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}))}\|u\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}))},$$

and that

$$\begin{aligned}\int_0^T \|u(t)\|^2_{\mathrm{L}^2(\mathbb{R})}dt &= 2\int_0^T \Re\left(-i\int_0^t \Phi h\overline{u}\; dxds\right)dt\\ &\leq 2T\|\Phi\|_c\|h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}))}\|u\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}))},\end{aligned}$$

thus,

$$\|h\|^2_{L^2(0,+\infty;L^2(\mathbb{R}))} \geq \frac{1-\gamma}{T\|\Phi\|_c^2}.$$

Second bound: The new boundary conditions $\|u(T)\|^2_{L^2(\mathbb{R})} < 4(1-\gamma)$ and $\|u_0\|^2_{L^2(\mathbb{R})} = 4$ give both that along with Cauchy Schwarz inequality

$$4\gamma < 2\|\Phi\|_c\|h\|_{L^2(0,+\infty;L^2(\mathbb{R}))}\|u\|_{L^2(0,T;L^2(\mathbb{R}))}$$

and also along with Cauchy Schwarz and integration over time

$$\|u\|^2_{L^2(0,T;L^2(\mathbb{R}))} - 4T \leq 2\|\Phi\|_c\|h\|_{L^2(0,+\infty;L^2(\mathbb{R}))}\|u\|_{L^2(0,T;L^2(\mathbb{R}))}.$$

Consequently, it follows that

$$\|u\|_{L^2(0,T;L^2(\mathbb{R}))} \leq T\|\Phi\|_c\|h\|_{L^2(0,T;L^2(\mathbb{R}))}\left(1+\sqrt{1+\frac{4}{T\|\Phi\|_c^2\|h\|^2_{L^2(0,T;L^2(\mathbb{R}))}}}\right).$$

Thus, we obtain

$$\frac{2\gamma}{T\|\Phi\|_c^2} < \|h\|^2_{L^2(0,+\infty;L^2(\mathbb{R}))}\left(1+\sqrt{1+\frac{4}{T\|\Phi\|_c^2\|h\|^2_{L^2(0,T;L^2(\mathbb{R}))}}}\right)$$

and

$$\|h\|^2_{L^2(0,T;L^2(\mathbb{R}))} > \frac{2(1+\gamma)}{T\|\Phi\|_c^2}\left(\sqrt{1+\left(\frac{\gamma}{1+\gamma}\right)^2}-1\right).$$

The upper bound follows. □

6.2. **Lower bounds.** We prove the following lower bounds.

Proposition 6.3. *For every positive T, γ in $[0,1]$, and every operator Φ in $\mathcal{L}_2(L^2(\mathbb{R}), H^1(\mathbb{R}))$ acting as the identity map on* $\mathrm{span}\{\frac{1}{\cosh(ax)}, x\frac{\sinh}{\cosh^2}(ax); a \in \mathbb{R}\}$, *the inequalities*

$$\underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}_\epsilon^{|0} \geq -\frac{2(1-\gamma)(12+\pi^2)}{9T}$$

and

$$\underline{\lim}_{\epsilon\to 0}\epsilon \log \mathbb{P}_\epsilon^{|1} \geq -\frac{2(2-\gamma-2\sqrt{1-\gamma})(12+\pi^2)}{9T}.$$

hold.

Proof. Let the constant η in the parametrized family of solitons depend on t and set

$$u(t,x) = \Psi_S(t,x) = \sqrt{2}\eta(t)\exp\left(-i\int_0^t \eta^2(s)ds\right)\mathrm{sech}(\eta(t)x) \tag{6.1}$$

then, from the assumption on Φ, the function

$$h_S(t,x) = i\frac{\eta'(t)}{\eta(t)}\Psi_S(t,x) - i\sqrt{2}\eta'(t)\eta(t)x\exp\left(-i\int_0^t \eta^2(s)ds\right)\frac{\sinh(\eta(t)x)}{\cosh^2(\eta(t)x)}$$

is such that $\mathbf{S}(h_S)$ and $\tilde{\mathbf{S}}(h_S)$ are the solutions of the control problems. Also, as u_0 belongs to $H^2(\mathbb{R})$, $\mathbf{S}(h_S)$ and $\tilde{\mathbf{S}}(h_S)$ are functions of $C([0,T];H^2(\mathbb{R}))\cap C^1([0,T];L^2(\mathbb{R}))$, consequently $t \to \eta(t) = \frac{1}{4}\|\Psi_S(t,\cdot)\|^2_{L^2(\mathbb{R})}$ is necessarily a function in $C^1([0,T])$.

For the first error probability, the lower bound follows from the fact that the infimum is smaller than the infimum on the smallest set of parametrized h_S and the computation of the $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}))$ norm of h_S which gives that

$$\inf_{\eta\in\ \mathrm{C}^1([0,T]):\eta(0)=0,\|\tilde{\mathbf{S}}(h_S)(T,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R})}dx\geq 4(1-\gamma)}\left\{\|h_S\|^2_{\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}))}\right\}$$

is equal to

$$\inf_{\eta\in\ \mathrm{C}^1([0,T]),b.c.}\int_0^T F_S(\eta(t),\eta'(t))dt,$$

where the Lagrangian F_S is

$$F_S(z,p)=\frac{1}{9}(12+\pi^2)\frac{p^2}{z},$$

and b.c. stands for the boundary conditions $\eta(0)=0$ and $\eta(T)\geq 1-\gamma$. Indeed, since $\tilde{\mathbf{S}}(h)(T)$ is a function of $(h(t))_{t\in[0,T]}$, the infimum could be taken on functions set to zero almost everywhere after T, thus $\|h\|^2_{\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}))}$ in the left hand side could be replaced by $\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}))}$. A scaling argument gives that the terminal boundary condition is necessarily saturated.
Similarly, for the second error probability, $\tilde{\mathbf{S}}$ is replaced by $\mathbf{S}$ and b.c. is $\eta(0)=1$ and $\eta(T)=1-\gamma$.
The usual results of the indirect method do not apply to the problem of the calculus of variations, nonetheless solutions of the boundary value problem associated to the Euler-Lagrange equation

$$2\frac{\eta''}{\eta}=\left(\frac{\eta'}{\eta}\right)^2$$

provide upper bounds when we compute the integral of the Lagrangian. If we suppose that η is in $\mathrm{C}^3([0,T])$ and that it is positive on $(0,T)$, we obtain by derivation of the ODE that on $(0,T)$,

$$\eta'''=0.$$

Also, looking for solutions of the form at^2+bt+c, we obtain that necessarily $b^2=4ac$. Thus $\mathrm{C}^3([0,T])$ positive solutions are necessarily of the form $a\left(t-\frac{b}{2a}\right)^2$. ¿From the boundary conditions, we obtain that for the first error probability the function defined by

$$\eta^0(t)=(1-\gamma)\left(\frac{t}{T}\right)^2$$

is a solution of the boundary value problem. For the second error probability, the boundary conditions imply that the two following functions defined by

$$\eta^{1,1}(t)=\left(2-\gamma+2\sqrt{1-\gamma}\right)\left(\frac{t}{T}\right)^2+2\left(-1-\sqrt{1-\gamma}\right)\frac{t}{T}+1$$

and

$$\eta^{1,2}(t)=\left(2-\gamma-2\sqrt{1-\gamma}\right)\left(\frac{t}{T}\right)^2+2\left(-1+\sqrt{1-\gamma}\right)\frac{t}{T}+1$$

are solutions of the boundary value problem. The second function gives the smallest value when we compute the integral of the Lagrangian. □

Remark that, in the case of the first error probability, both upper and lower bounds in Proposition 6.2 and Proposition 6.3 are increasing functions of γ. Similarly, in the case of the second error probability, the bounds are decreasing functions of γ. This could be interpreted as the higher is the threshold, the more energy is needed to form a signal which momentum gets above the threshold at the coordinate T and conversely in the case of a soliton as initial datum.

Remark 6.4. When there is no particular tradeoff between the two errors, the overall risk of error in transmission, due to noise, can be taken as

$$R_\epsilon = \mathbb{P}_\epsilon^{|0} \vee \mathbb{P}_\epsilon^{|1}.$$

Choosing $\gamma = \frac{5}{7}$ allows to minimize the maximum of the two upper bounds of Proposition 6.2, for the associated threshold we get

$$\overline{\lim}_{\epsilon\to 0}\epsilon \log R_\epsilon \leq -\frac{1}{7T\|\Phi\|_c^2}.$$

Similarly, choosing $\gamma = \frac{3}{4}$ minimizes the maximum of the two lower bounds of Proposition 6.3, for the associated threshold we get

$$-\frac{12+\pi^2}{18T} \leq \underline{\lim}_{\epsilon\to 0}\epsilon \log R_\epsilon.$$

The two values of γ are very close and correspond to thresholds taken as 1.14 and 1 respectively, i.e. to 22% and 25% of the momentum of the initial datum.

Remark also that the bounds for the error probabilities are of the right order. Indeed, from the probability density function given for the first error probability in [16], we are expecting, when the noise is the ideal white noise and thus $\|\Phi\|_c = 1$, that $\lim_{\epsilon\to 0}\epsilon \log \mathbb{P}_\epsilon^{|0} = -c\frac{1-\gamma}{T}$ with a positive constant c. The joint probability density function obtained in [16], when the initial datum is a soliton, also allows to obtain asymptotics of the probability of tail events of the soliton power and thus of the second error probability.

Remark finally that it is natural to obtain that the opposite of the error probabilities are decreasing functions of T. Indeed, the higher is T, the less energy is needed to form a signal which momentum gets above a fixed threshold at the coordinate T. Replacing above by under, we obtain the same result in the case of a soliton as initial datum. Consequently, the higher is T the higher the error probabilities get.

We give in the next section some further considerations on problems of the calculus of variations along with the results of some numerical computations.

6.3. **Remarks on the problem of the calculus of variations.** The most efficient parametrization we obtained has been presented above. Many parametrizations are at hand and we may expect one that allows to obtain a good approximation of the rate of convergence to zero of the error probabilities. This would allow to have an idea of the most likely trajectories leading to an error in transmission. The presence of an operator Φ before h in the control problem is a limitation for the computation. We will thereafter consider that Φ acts as the identity map on a sufficiently large linear space of $\mathrm{L}^2(\mathbb{R})$.

A parametrization as a function of t and x. If we look for solutions of the form $u_0\left(1-\frac{t}{T}\right)+u_T\frac{t}{T}$ and suppose that u_T belong to $\mathrm{H}^3(\mathbb{R}^d)$, the infimum over the functions u_T gives birth to a Euler-Lagrange equation which is a fourth order nonlinear PDE. A more reasonable approach for computation is to find an upper bound as an infimum functions parametrized by certain paths on $\mathbb{R}$.

The parametrization by the amplitude. We consider paths of the form $u(t,x)=f(t)u_0(x)$ for $f\in\mathrm{H}^1(0,T)$. In that case $\mathrm{H}^1(0,T)$ is a space of real valued functions. We recall that, from the Sobolev inequalities, $\mathrm{H}^1(0,T)$ is continuously embedded into the space of $\frac{1}{2}-$Hölder functions $\mathrm{C}^{\frac{1}{2}}([0,T])$ and thus into every $\mathrm{L}^p(0,T)$. Remark that the function $h_a(t)=if'(t)u_0-f(t)\Delta u_0-f(t)^3|u_0|^2u_0$ is the associated control. Minimizing over the norm of these controls leads to a problem of the calculus of variations. We obtain that

$$L^{(t,+\infty)}\leq\frac{1}{2}\inf_{f\in\mathcal{A}}\int_0^T F_a(f(t),f'(t))dt$$

where the Lagrangian F_a is defined by

$$F_a(z,p)=p^2\|u_0\|^2_{\mathrm{L}^2(\mathbb{R})}+z^2\|\Delta u_0\|^2_{\mathrm{L}^2(\mathbb{R})}+2z^4\left(\Delta u_0,|u_0|^2u_0\right)_{\mathrm{L}^2(\mathbb{R})}+z^6\||u_0|^2u_0\|^2_{\mathrm{L}^2(\mathbb{R})}$$

which, with the particular value of u_0, becomes

$$F_a(z,p)=4p^2+\frac{28}{15}z^2-\frac{32}{5}z^4+\frac{128}{15}z^6,$$

and $\mathcal{A}$ is the admissible set $\{f\in\mathrm{H}^1(0,T):f(0)=0,\ f(T)=\sqrt{1-\gamma}\}$ for the first error probability and $\{f\in\mathrm{H}^1(0,T):f(0)=1,\ f(T)=\sqrt{1-\gamma}\}$ for the second error probability.
Indeed, as for every $\alpha\geq 1$, $\frac{28}{15}(\alpha^2-1)z^2-\frac{32}{5}(\alpha^4-1)z^4+\frac{128}{15}(\alpha^6-1)z^6\geq 0$, we have $F_a(\alpha z,\alpha p)\geq F_a(z,p)$, thus the terminal boundary condition corresponding to the first error probability is necessarily saturated. The same holds for the terminal value corresponding to the second error probability changing z to $z-1$.

Also, standard calculation gives that for every (z,p) in $\mathbb{R}^2$, $F_a(z,p)\geq 4p^2+\frac{2}{3}z^2$. Recall that as F_a is non-negative, convex in the p variable, F_a and $\frac{\partial}{\partial p}F_a$ are continuous in the (z,p) variables, the integral is a weakly sequentially lower semicontinuous function of f. In addition, as in both cases $\mathcal{A}$ is nonempty and F_a satisfies the coercivity condition $F_a(z,p)\geq 4p^2$ for every (z,p) in $\mathbb{R}^2$, there exists at least one minimizer, i.e. a function that solves the problem of the calculus of variations.

It is finally also possible to revisit slightly the proof of Proposition 4 of section 8.2.3 of reference [15] and to check that, in our particular case, though we do not have the expected growth conditions on the Lagrangian, Lebesgue's dominated convergence theorem could be applied and that the result of the proposition still holds. Consequently, any minimizer is a weak solution of the Euler-Lagrange equation. If we apply the chain rule supposing that f is a function of $\mathrm{H}^2(0,T)$, the Euler-Lagrange equation becomes the following nonlinear ODE

$$15f''-7f+48f^3-96f^5=0.$$

By numerical computations of the integral and of the boundary value problems for the first and second error probabilities, we obtain in figure 1 the curves labeled "amplitude". These bounds are compared to the bounds of section 6.2 where the curves are labeled "soliton parameter" and to the upper bounds of section 6.1.

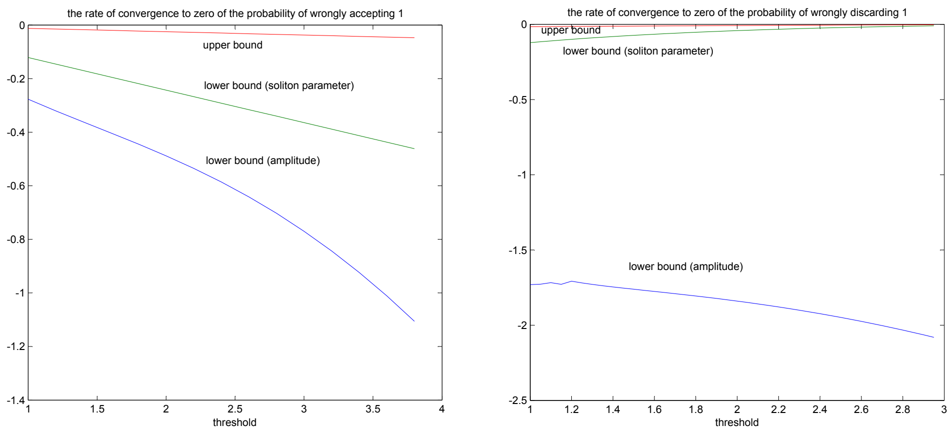

FIGURE 1. Bounds as a function of the threshold $4(1-\gamma)$ with $T = 10$ and $\|\Phi\|_c = 1$.

As we may see in figure 1, this parametrization is less efficient to obtain lower bounds. Moreover, the case of interest in fibers is when T is large and a graph of T times the integral shows that this parametrization gives a lower bound of order less than $\frac{1}{T}$. It is not satisfactory, nonetheless it is presented in the paper because it could be used with arbitrary u_0, σ and d, to obtain some refinement of the result of Proposition 5.3. Remark also that in this case a nice parametrization could give an idea of the most likely trajectories of the solutions that blow up after time T.

These computations show that a parametrization of the amplitude of the soliton is not enough to approximate the most likely trajectories leading to an error in transmissions. The phase and several other shape parameters have to be introduced. Remark that we also tried a parametrization replacing in the exponential in the identity (6.1) the term $\int_0^t \eta^2(s)ds$ by $\eta^2(t)t$ but that it gave less interesting results.

A more complete parametrization. In reference [16] the authors parametrize the signal in the following way

$$u(t,x) = \Psi_p(t,x) = \sqrt{2}\eta(t)\exp\left(i\beta(t)x + i\alpha(t) + i\tau(t)\right)\left[\operatorname{sech}(\eta(t)(x-y(t))) + v(x)\right],$$

where τ is the "internal time", it satisfies $\tau'(t) = \eta^2(t)$, and v the "continuous spectrum on the background of the soliton". This parametrization is used to obtain a good approximation of the probability density function of the joint law of the soliton power and of the shift of the soliton position. We saw that introducing a field v strongly complicates the calculus of variations. An interested reader could refer to the article for a physical justification that v is indeed negligible in first approximation. The authors also neglect the α variable, we keep it. The associated control is the function

$$h_p(t,x) = i\frac{\eta'(t)}{\eta(t)}\Psi_p(t,x) - (\beta'(t)x + \alpha'(t))\Psi_p(t,x)$$
$$+\sqrt{2}i[-\eta'(t)(x-y(t)) + \eta(t)y'(t)]\exp\left(i\beta(t)x + i\alpha(t) + i\tau(t)\right)\eta(t)\frac{\sinh}{\cosh^2}(\eta(t)(x-y(t))).$$

The augmented Lagrangian of the new problem of the calculus of variations is given by

$$F_P(Z,P) = \frac{12+\pi^2}{9}\frac{p_1^2}{z_1} + \frac{4}{3}z_1^3p_4^2 + 4p_2^2z_1 + 4p_3^2z_1z_4^2 + \frac{\pi^2p_3^2}{3z_1} + 8p_2p_3z_1z_4,$$

where $Z = (z_1,z_2,z_3,z_4) = (\eta,\alpha,\beta,y)$ and $P = (p_1,p_2,p_3,p_4)$ is its derivative. A scaling argument still indicates that the terminal boundary conditions are saturated, thus $\eta(0) = 0$ and $\eta(T) = 1-\gamma$ in the case of the first error probability and $\eta(0) = 1,\ \alpha(0) = \beta(0) = y(0) = 0$ and $\eta(T) = 1-\gamma$ for the second error probability. Still, the usual results of the indirect method do not apply and we are not assured that a minimizer is a solution of the Euler-Lagrange nor of the contrary. Nonetheless, the solutions of the Euler-Lagrange equation may be good guesses and provide upper bounds of the problem of the calculus of variations and thus lower bounds of the rate of exponential convergence to zero of the error probabilities. It consists of a system of coupled nonlinear ODEs. The natural boundary conditions at the free end leads to $\alpha' = \beta' = 0$ and the system simplifies in

$$\left\{\begin{array}{c} \frac{12+\pi^2}{9}\left(\frac{2\eta''}{\eta} - \left(\frac{\eta'}{\eta}\right)^2\right) = 4\eta^2(y')^2 \\ \eta'\eta^2y' + \frac{\eta^3y''}{3} = 0 \end{array}\right.$$

with the extra condition $y'(T) = 0$ resulting from the natural boundary conditions. It is associated to the simpler Lagrangian

$$F_{PS}(z_1,z_4,p_1,p_4) = \frac{12+\pi^2}{9}\frac{p_1^2}{z_1} + \frac{4}{3}z_1^3p_4^2.$$

The singularity did not allow to treat numerically the case of null initial datum and led to an identically null solution for y' and thus to the same solution for η and the same lower bound as in section 6.2 in the case of a soliton as initial datum.

The parametrization is probably very well adapted to the case of a bounded window, particularly for the second error probability. The variable y is a parametrization of the shift of the soliton position, it is also likely to be relevant for the computation of a lower bound by the calculus of variations in the next section.

6.4. **The tails of the law of the shift of the soliton position when ϵ converges to zero.** We consider the mapping Y from $\Sigma^{\frac{1}{2}}$ into $\mathbb{R}$ defined by

$$Y(f) = \int_{\mathbb{R}} x|f(x)|^2dx.$$

We suppose that Φ belongs to $\mathcal{L}_2(\mathrm{L}^2(\mathbb{R}^d),\Sigma)$ and that $\ker\Phi^* = \{0\}$. The following lemma allows to define the family of measures $\left(\mu^{Y_\epsilon} = (Y \circ eval_T \circ \mathcal{G})_*\mu^{Z_\epsilon}\right)_{\epsilon>0}$ which correspond to the laws of the shift of the position of the soliton for each solution u_ϵ with a soliton as initial datum and for a noise of intensity ϵ. Also, since the shift is zero for the deterministic solution u_d, the lemma gives that the sequence of measures converges weakly to the Dirac mass on 0. Proposition 6.6 characterizes the convergence to zero of the tails.

Lemma 6.5. *For every positive T, the mapping $Y \circ eval_T \circ \mathcal{G}$ from $X^{(T,4);\Sigma} = \mathrm{C}([0,T];\Sigma) \cap \mathrm{L}^{r(4)}\left(0,T;\mathrm{W}^{1,4}(\mathbb{R})\right)$ into $\mathbb{R}$ is continuous.*

Proof. Let Z and Z' belong to $X^{(T,4);\Sigma}$, the triangular inequality along with Hölder's inequality allow to compute the following sequence of inequalities

$$\begin{aligned}
&\left|\int_{\mathbb{R}} x\left(|\mathcal{G}(Z)(T,x)|^2 - |\mathcal{G}(Z')(T,x)|^2\right) dx\right| \\
&\leq \int_{\mathbb{R}} |x|(|\mathcal{G}(Z)(T,x)| + |\mathcal{G}(Z)(T,x)|)|(|\mathcal{G}(Z)(T,x)| - |\mathcal{G}(Z)(T,x)|)|dx \\
&\leq \|\mathcal{G}(Z)(T) - \mathcal{G}(Z')(T)\|_{\mathrm{L}^2(\mathbb{R})}^{\frac{1}{2}} \sqrt{V(|\mathcal{G}(Z)(T)| + |\mathcal{G}(Z')(T)|)},
\end{aligned}$$

where V is the variance , it is defined in section 2.
After some calculations we obtain that

$$\sqrt{V(|\mathcal{G}(Z)(T)| + |\mathcal{G}(Z')(T)|)}$$

is lower than

$$2\sqrt{2}\left(\sqrt{V(v(Z)(T))} + \sqrt{V(v(Z')(T))} + \sqrt{V(Z(T))} + \sqrt{V(Z'(T))}\right).$$

The application of Gronwall's inequality given in the proof of Proposition 3.5 of reference [10], along with the Sobolev injection of $\mathrm{H}^1(\mathbb{R})$ into $\mathrm{L}^\infty(\mathbb{R})$ and the continuity of $\mathcal{G}$ from $X^{(T,4);\Sigma}$ into $\mathrm{C}([0,T];\mathrm{H}^1(\mathbb{R}))$ give that this last term is bounded when Z and Z' are in a bounded set. We conclude using the continuity of $\mathcal{G}$. □

The fact that Z defines a $X^{(T,4);\Sigma}$ random variable follows from similar arguments as those used in the proof of Proposition 2.2. We still denote by μ^Z its law and by μ^{Z_ϵ} the direct images under the transformation $x \mapsto \sqrt{\epsilon}x$ on $X^{(T,4);\Sigma}$.

Proposition 6.6. *The family of measures (μ^{Y_ϵ}) on $\mathbb{R}$ satisfy a LDP of speed ϵ and good rate function*

$$I^Y(y) = \frac{1}{2} \inf_{h \in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R})): \int_{\mathbb{R}} x|\mathbf{S}(h)(T,x)|^2 dx = y} \left\{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}))}\right\}.$$

Moreover for every nonempty interval J such that $0 \notin J$, $0 < \inf_{y \in J} I^Y(y) < \infty$, and for every positive R besides an at most countable set of points, the following equivalents hold

$$\begin{aligned}
\log \mathbb{P}(Y_\epsilon \geq R) &\sim_{\epsilon \to 0} -\frac{1}{2\epsilon} \inf_{h \in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R})): \int_{\mathbb{R}} x|\mathbf{S}(h)(T,x)|^2 dx \geq R} \left\{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}))}\right\} \\
\log \mathbb{P}(Y_\epsilon \leq -R) &\sim_{\epsilon \to 0} -\frac{1}{2\epsilon} \inf_{h \in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R})): \int_{\mathbb{R}} x|\mathbf{S}(h)(T,x)|^2 dx \leq -R} \left\{\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}))}\right\}.
\end{aligned}$$

Proof. The LDP for the family $(\mu^{Z_\epsilon})_{\epsilon>0}$, which are centered gaussian measures on a real Banach space, the fact that their RKHS is $\mathrm{im}\mathcal{L}$ with the norm of the image structure and Varadhan's contraction principle give that the family $\left(\mu^{Y_\epsilon}\right)_{\epsilon>0}$

satisfy a LDP of speed ϵ and good rate function which is the rate function of the proposition

$$I^Y(y) = \inf_{z\in X^{(T,4);\Sigma}:Y\circ\ eval_T\circ\ \mathcal{G}(z)=y}\left\{\inf_{h\in \mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R})):\mathcal{L}(h)=z}\left\{\frac{1}{2}\|h\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}))}\right\}\right\}.$$

The fact that $\inf_{y\in J} I^Y(y) < \infty$ follows from the assumption $\ker\Phi^* = \{0\}$ and that for every real number a, a solution of the form $u(t,x) = (1+atx)u_0$ satisfies $Y(u(T)) = \frac{aT\pi^2}{3}$, i.e. there exists controls such that the solution reaches any interval at "time" T.
The positivity follows from the continuity of the mapping $Y\circ eval_T\circ\mathcal{G}\circ\Lambda\circ\Phi$ from $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}))$ into $\mathbb{R}$. Indeed Φ is continuous from $\mathrm{L}^2(0,T;\mathrm{L}^2(\mathbb{R}))$ into $\mathrm{L}^1(0,T;\Sigma)$ and Λ is continuous from $\mathrm{L}^1(0,T;\Sigma)$ into $X^{(T,4);\Sigma}$. This last statement follows from the Strichartz estimates, the facts that the group is a unitary group on Σ and that the integral is a linear continuous mapping from $\mathrm{L}^1(0,T;\Sigma)$ into Σ.
The proofs of the final statements are the same as those of Proposition 6.1. $\square$

Remark that the results are true in $\mathbb{R}^d$ with an arbitrary order of subcritical nonlinearity.

6.5. **The transmission in the general case where blow-up may occur.** We are interested by estimates of the probabilities

$$\mathbb{P}\left(\int_{-l}^{l}|u_\epsilon^n(T,x)|^2dx \geq 4-\gamma, \mathcal{T}(u_\epsilon^n) > T\right) = \mathbb{P}_\epsilon^{|0,\mathcal{E}}$$

and of

$$\mathbb{P}\left(\int_{-l}^{l}|u_\epsilon(T,x)|^2dx < 4-\gamma, \mathcal{T}(u_\epsilon) > T\right) = \mathbb{P}_\epsilon^{|1,\mathcal{E}}.$$

Proposition 6.7. *The following inequalities for the two error probabilities hold,*

$$-L^{|0,\mathcal{E}} \leq \underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}_\epsilon^{|0,\mathcal{E}} \leq \overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}_\epsilon^{|0,\mathcal{E}} \leq -U^{|0,\mathcal{E}}$$

where

$$L^{|0,\mathcal{E}} = \frac{1}{2}\inf_{h\in\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d)):\int_{-l}^{l}|\tilde{\mathbf{S}}(h)(T,x)|^2dx>4-\gamma,\mathcal{T}(\tilde{\mathbf{S}}(h))>t}\left\{\|h\|^2_{\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d))}\right\}$$

and

$$U^{|0,\mathcal{E}} = \frac{1}{2}\inf_{h\in\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d)):\int_{-l}^{l}|\tilde{\mathbf{S}}(h)(T,x)|^2dx\geq 4-\gamma}\left\{\|h\|^2_{\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d))}\right\} < 0,$$

similarly,

$$-L^{|1,\mathcal{E}} \leq \underline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}_\epsilon^{|1,\mathcal{E}} \leq \overline{\lim}_{\epsilon\to 0}\epsilon\log\mathbb{P}_\epsilon^{|1,\mathcal{E}} \leq -U^{|1,\mathcal{E}}$$

where

$$L^{|1,\mathcal{E}} = \frac{1}{2}\inf_{h\in\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d)):\int_{-l}^{l}|\tilde{\mathbf{S}}(h)(T,x)|^2dx<4-\gamma,\mathcal{T}(\mathbf{S}(h))>t}\left\{\|h\|^2_{\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d))}\right\}$$

and

$$U^{|1,\mathcal{E}} = \frac{1}{2}\inf_{h\in\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d)):\int_{-l}^{l}|\mathbf{S}(h)(T,x)|^2dx\leq 4-\gamma}\left\{\|h\|^2_{\mathrm{L}^2(0,+\infty;\mathrm{L}^2(\mathbb{R}^d))}\right\} < 0.$$

Proof. The result follows from the LDP for the laws of the solutions $(u_\epsilon^n)_{\epsilon>0}$, the fact that $\varphi^{-1}([4-\gamma,+\infty))$ is a closed set containing $\varphi^{-1}([4-\gamma,+\infty))\cap \mathcal{T}^{-1}((T,+\infty))$ and that $\varphi^{-1}((4-\gamma,+\infty))\cap\mathcal{T}^{-1}((T,+\infty))$ is an open set included in $\varphi^{-1}([4-\gamma,+\infty))\cap\mathcal{T}^{-1}((T,+\infty))$. $\square$

Remark again that, under the assumption that the null space of Φ^* is zero, the infima are never taken over empty sets. If l is taken as $+\infty$, we obtain the same upper bounds as in section 6.1. Also, if Φ acts as the identity on a sufficiently large linear space of $\mathrm{L}^2(\mathbb{R}^d)$, we could implement the previous computations of the calculus of variations on the parametrized controls.

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